Local heights computations for quadratic Chabauty

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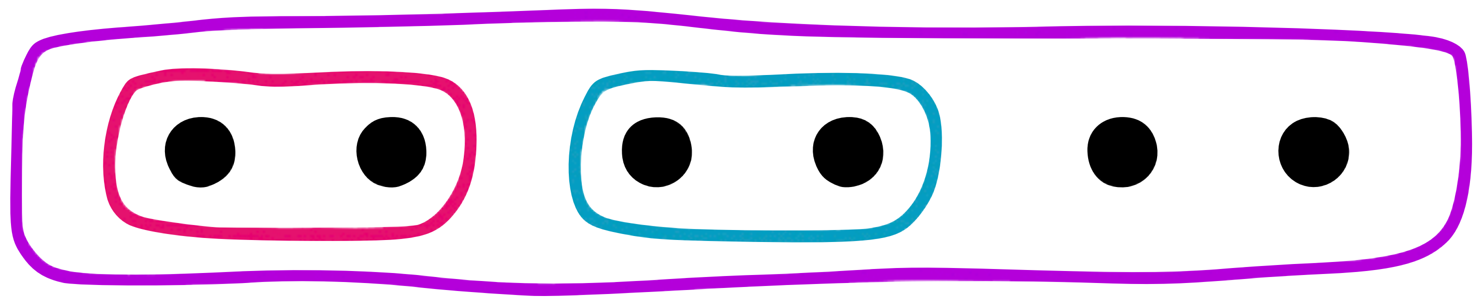
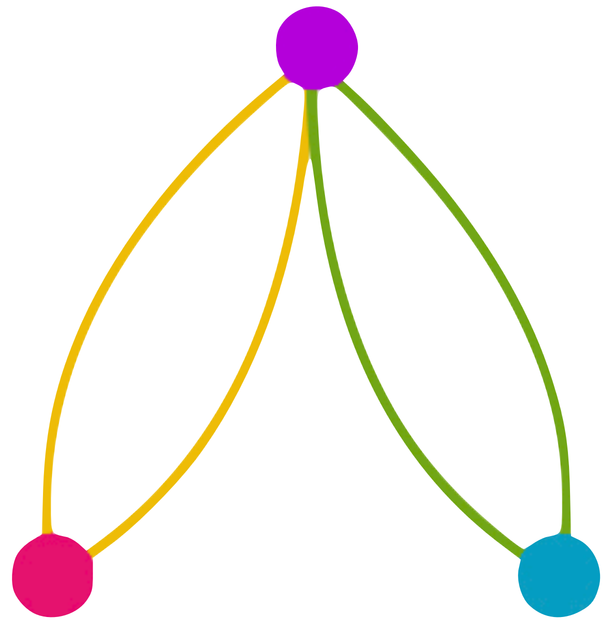
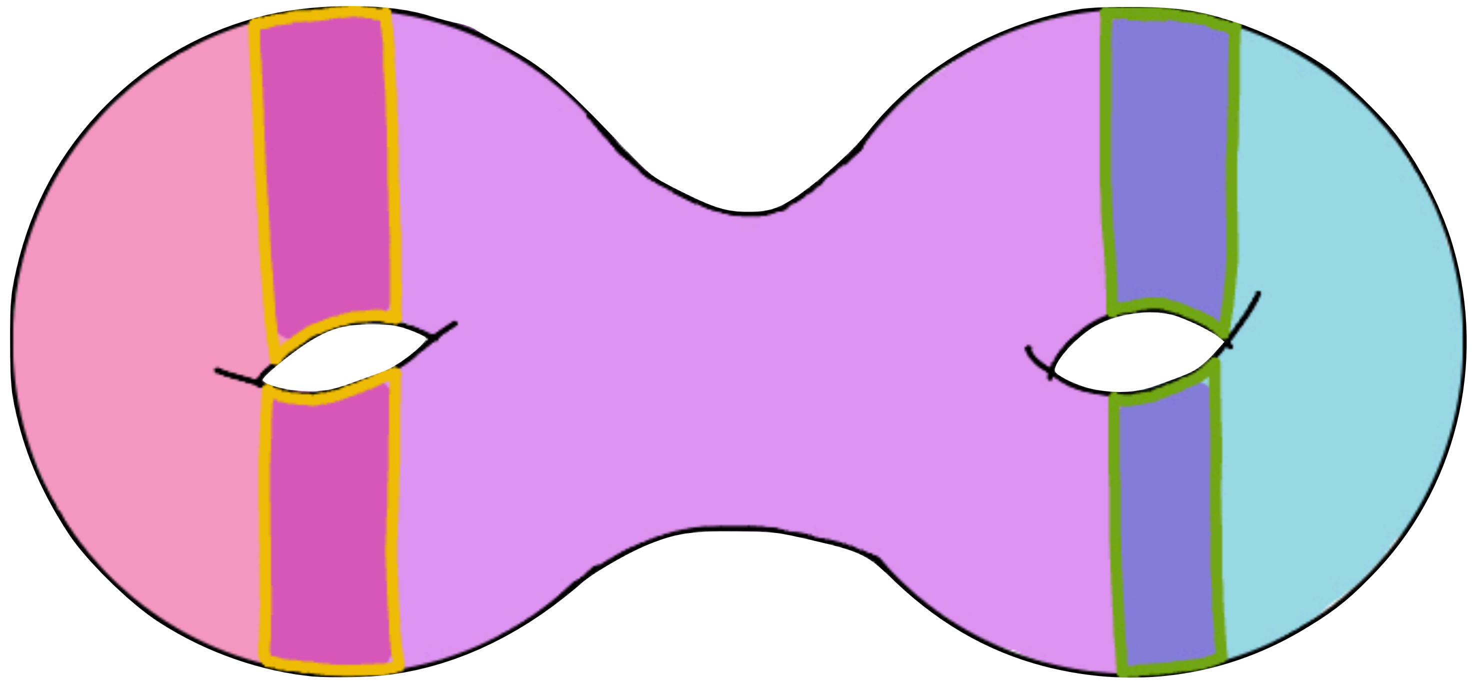
Joint work with Alexander Betts, Sachi Hashimoto, and Pim Spelier.

Rational points on curves

For this talk, $C/\mathbb{Q}$ will be a nice curve of genus $g \geq 2$.

Theorem [Faltings, '83]. $\#C(\mathbb{Q}) < \infty$.

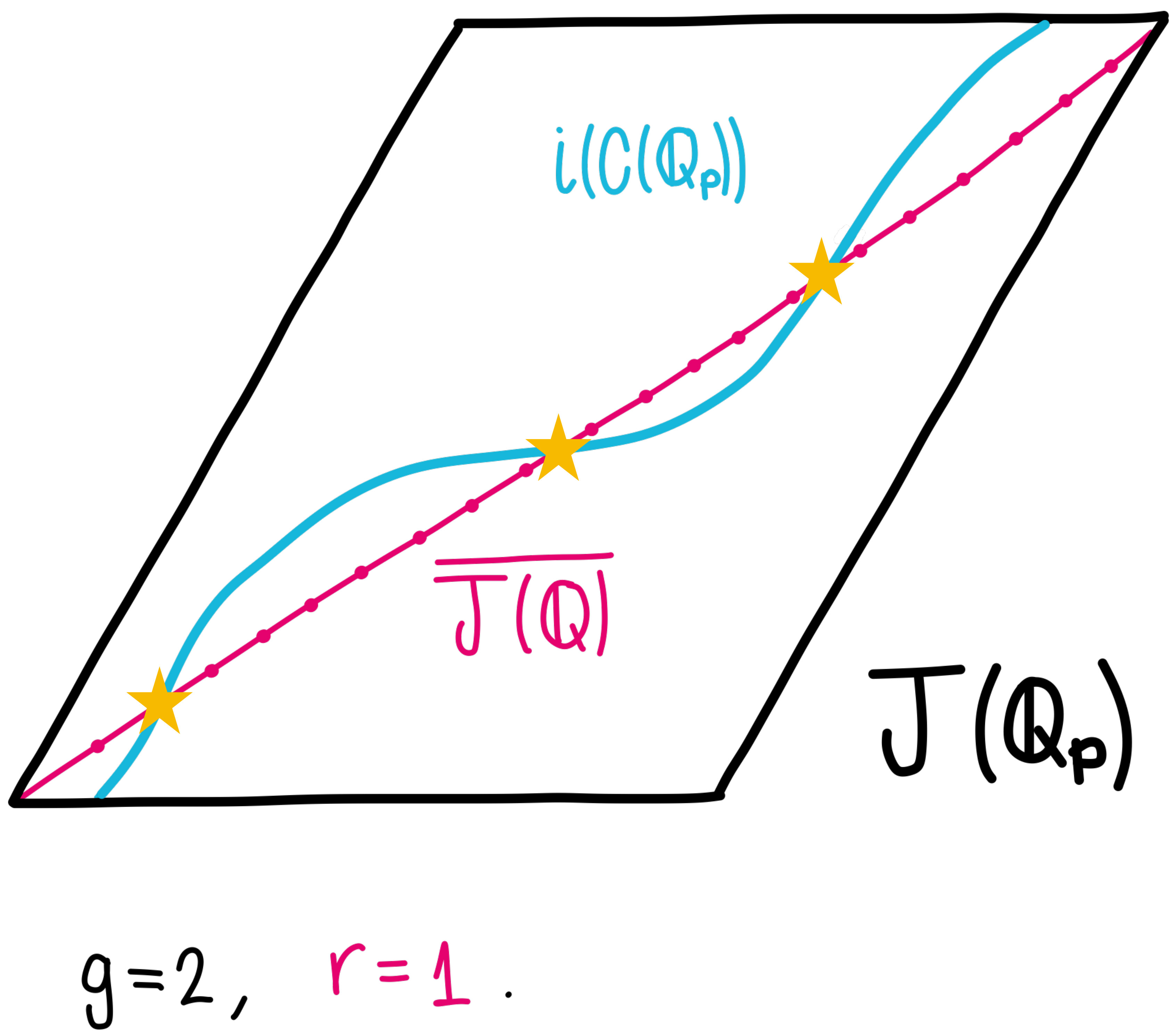
But how do we make it effective?

- $J := \mathrm{Jac}(C)$.
- r is the Mordell-Weil rank of J.
- p is a prime number.

Theorem [Chabauty, '41]. If $r < g$, then

$$\iota(C(\mathbb{Q})) \subseteq \iota(C(\mathbb{Q}_p)) \cap \overline{J(\mathbb{Q})} \subseteq J(\mathbb{Q}_p),$$

and this intersection is finite.



Quadratic Chabauty

Chabauty—Kim's Method ['05, '09]. Goal is to use p-adic methods to determine $C(\mathbb{Q})$.

Balakrishnan—Dogra ['18, '20]. Chabauty—Kim's Program is made explicit for $r = g$ and p of good reduction.

Theorem [Balakrishnan—Dogra, '18]. Suppose that $r = g$ $(+\epsilon)$, and that $Z \subset C \times C$ is a trace 0 correspondence. Then there exists a quadratic function

$$Q_Z\colon \operatorname{Lie}\left(J_{\mathbb{Q}_p}\right) \to \mathbb{Q}_p$$

for which $C(\mathbb{Q})$ is contained in the locus inside $C(\mathbb{Q}_p)$ cut out by the equation

$$Q_Z(\log(z)) - h_{Z,p}(z) \in \Omega,$$

where $\Omega = \left\{ \sum_{\ell \neq p} h_{Z,\ell}(x_\ell) \ : \ x_\ell \in C(\mathbb{Q}_\ell) \right\}$.

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Quadratic Chabauty and heights

Key input: Let p be a prime number and $Z \subset C \times C$ be a trace 0 correspondence. There the associated p-**adic** (Coleman—Gross) **height function** $h_Z : C(\mathbb{Q}) \to \mathbb{Q}_p$ can be decomposed as

$$h_Z(Q) = \sum_{\ell} h_{Z,\ell}(Q),$$

where $h_{Z,\ell} : C(\mathbb{Q}_\ell) \to \mathbb{Q}_p$.

- For $\ell \neq p$ the height function $h_{Z,\ell}(x_\ell)$ takes only finitely many values for $x_\ell \in C(\mathbb{Q}_\ell)$.
- If $\ell \neq p$ is a prime of potential good reduction, then $h_{Z,\ell}(x_\ell) = 0$ for all $x_\ell \in C(\mathbb{Q}_\ell)$.
- There is no known general explicit algorithm to compute $h_{Z,\ell}$. Quadratic Chabauty computations are done in a case-by-case basis.

Today's problem: **to compute heights $h_{Z,\ell}$ for $\ell \neq p$ of bad reduction.**

Explicit height computations

Theorem [Betts—DR.—Hashimoto—Spelier, '23]. Let $C/\mathbb{Q}$ be a hyperelliptic curve that admits a model $y^2 = f(x)$, where $f(x)$ is separable and of degree ≥ 3. Let $Z \subset C \times C$ be a trace 0 correspondence. Let p be a prime number. Then there is a provably correct algorithm to compute the function $h_{Z,\ell}$ for all odd primes of bad reduction $\ell \neq p$ and $\ell \neq 2$.

Today's problem: **to compute heights $h_{Z,\ell}$ for $\ell \neq p$ of bad reduction**.

Explicit height computations: how?

Assume for simplicity that $C/\mathbb{Q}_\ell$ has semistable reduction and that all components of its special fibre have genus 0.

Theorem [Betts–Dogra, '20]. Let p be a prime number and $Z \subset C \times C$ be a trace 0 correspondence. Then, there is an explicit formula for computing $h_{Z,\ell}$ in terms of the induced action of Z_* on the homology $\mathrm{H}_1(\Gamma, \mathbb{Q})$ of the dual graph Γ of the geometric special fibre. This formula uses a semistable model of C.

Today's problem: to compute **the action of Z_* on** $\mathrm{H}_1(\Gamma, \mathbb{Q})$ (without computing the reduction of Z!)

Coleman - Iovita

We can represent Z as a divisor in $C \times C$ [Costa—Mascot—Sijsling—Voight, '19].

This allows us to understand the action of Z_* on $\mathrm{H}^0(C_{\mathbb{Q}_\ell}, \Omega^1_C)$.

Theorem [Coleman & Iovita, '99]. The map

$$\mathrm{H}^0(C_{\mathbb{Q}_\ell}, \Omega^1_C) \to \mathrm{H}_1(\Gamma, \mathbb{Q}_\ell) \quad \text{given by} \quad \omega \mapsto \sum_{e \in E(\Gamma)} \mathrm{Res}_{A_{\vec{e}}}(\omega) \cdot \vec{e}$$

is surjective. Moreover, the map is an isomorphism if every component of the ℓ-adic special fibre has genus 0.

We can compute the action of Z_* on $\mathrm{H}_1(\Gamma, \mathbb{Q}_\ell)$ up to any desired ℓ-adic precision.

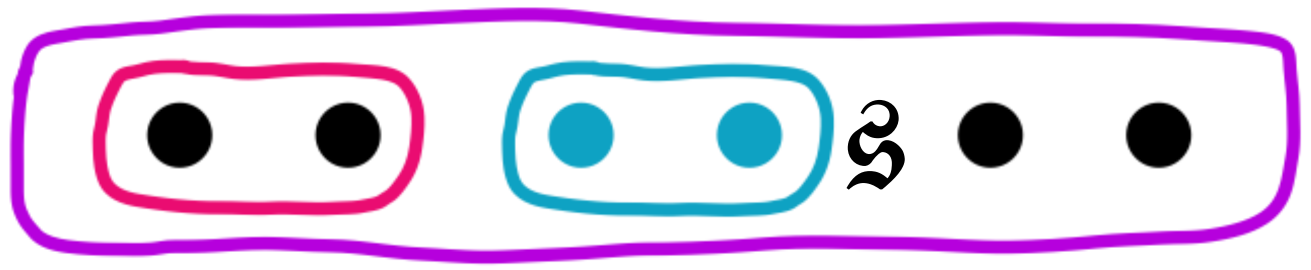
Coleman - Iovita done explicitly

Theorem [Betts—DR—Hashimoto—Spelier, '23]. The endomorphism of $\mathrm{H}_1(\Gamma, \mathbb{Q})$ given by Z_* is defined over $\mathbb{Z}$, and has operator norm $\leq \sqrt{d_1 d_2}$, where d_1 and d_2 are the degrees of the two projections $Z \to C$.

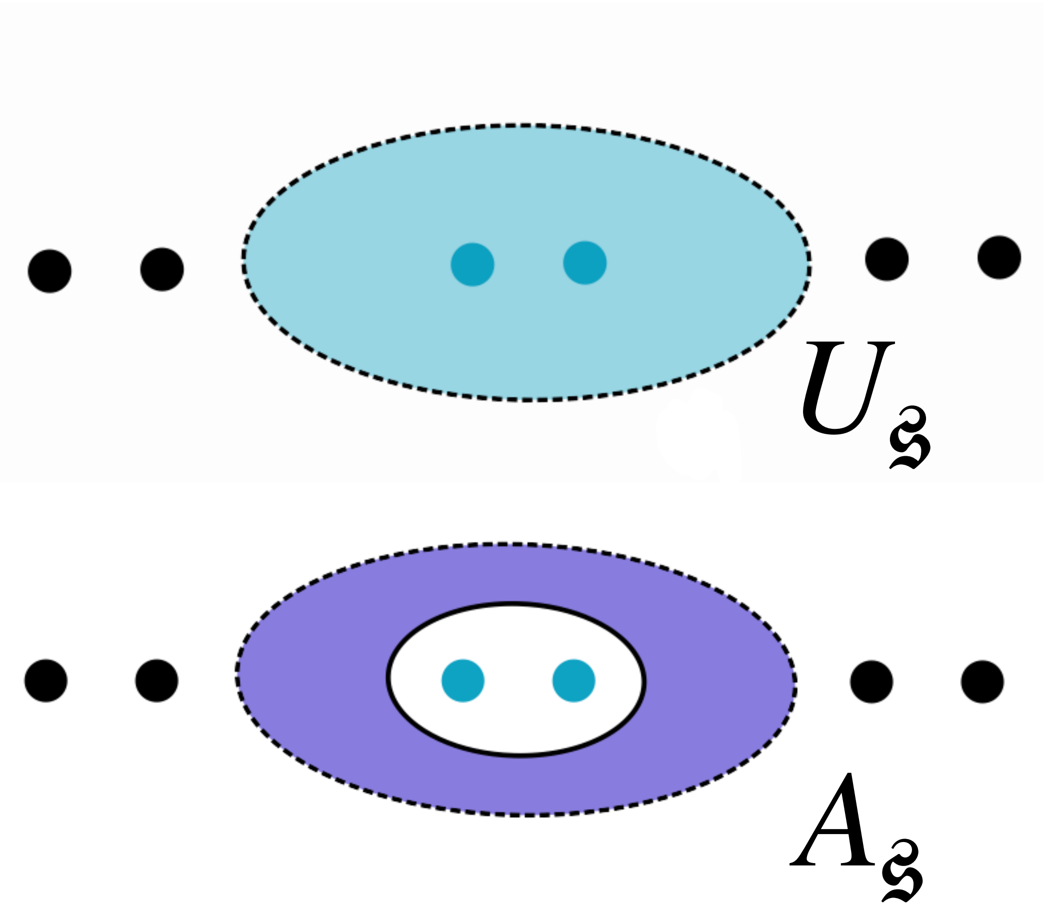
Finite precision is enough!

A semistable covering

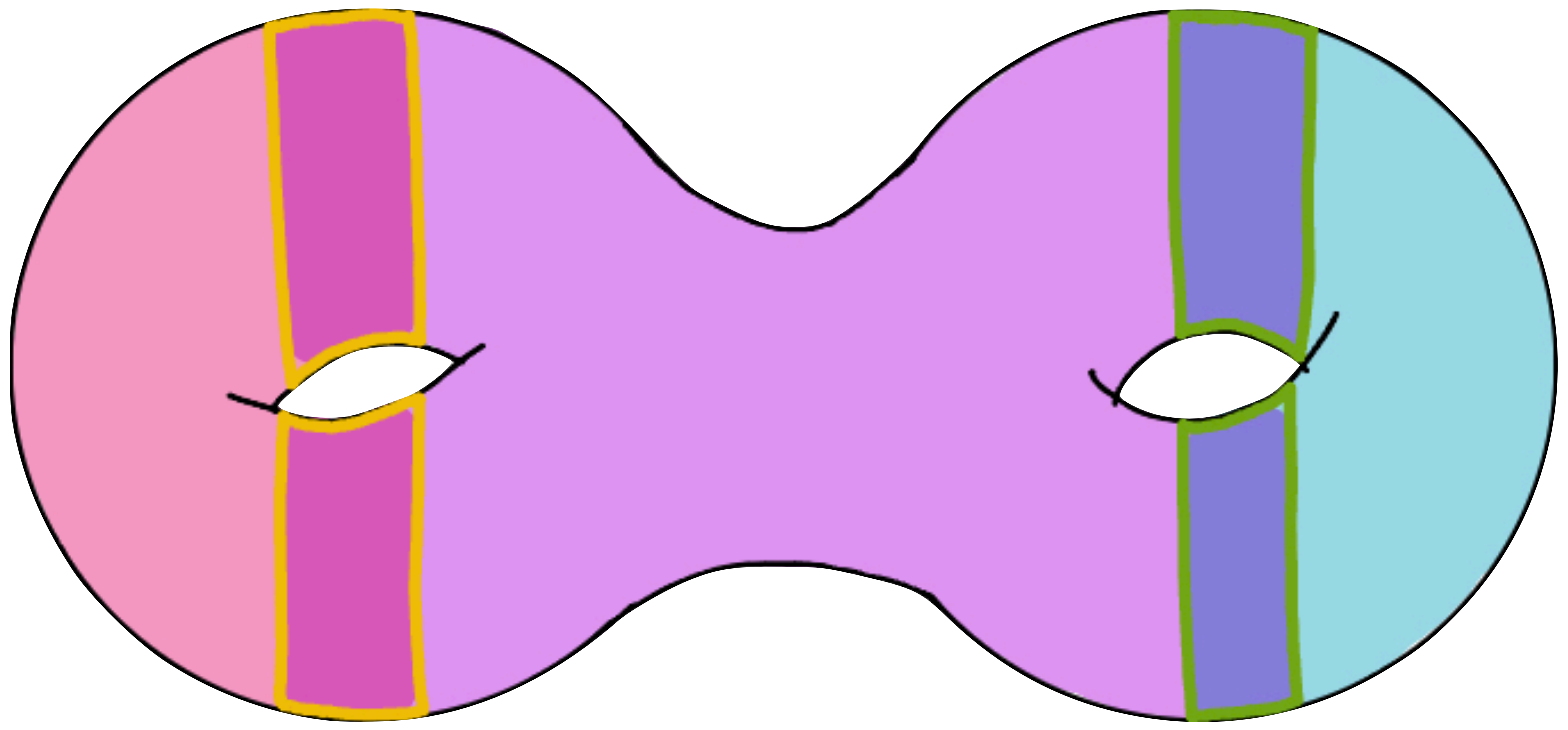
In general, we cannot compute semistable models of C (or work explicitly with them).
Instead, we construct a semistable covering of C using cluster pictures [Best et al., '22].



Cluster picture:
roots of $f(x)$.



Semistable
covering of $\mathbb{P}^{1,\mathrm{an}}$

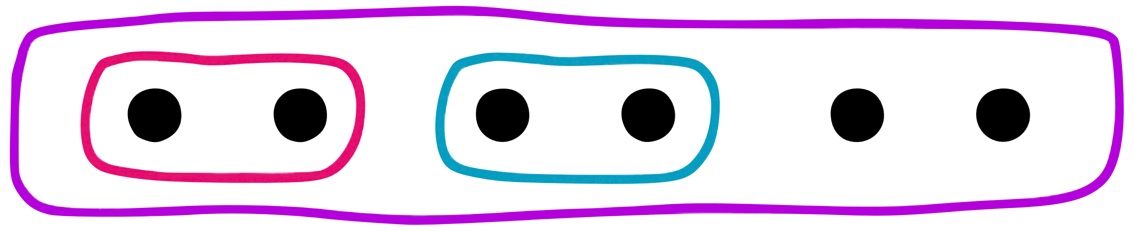


Semistable
covering of C^{an}

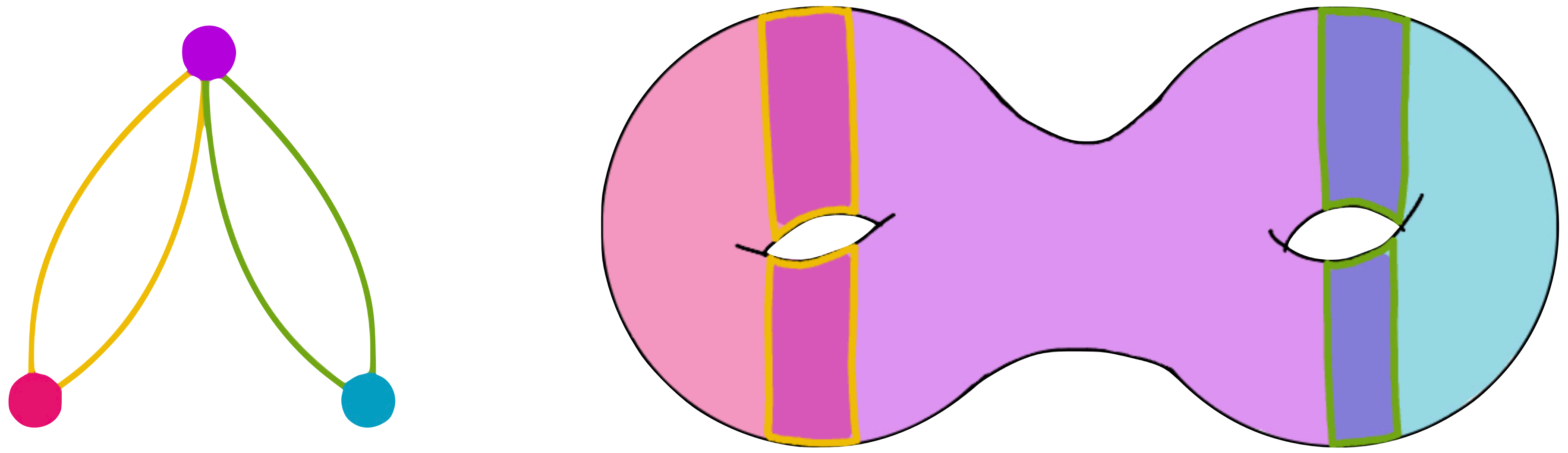
Local heights computations: an example

$$C \ : \ y^2 = x^6 + 2x^4 + 6x^3 + 5x^2 - 6x + 1$$

We pick a correspondence $Z \subset C \times C$ with action on $H^0(X, \Omega^1_X)$ given by $\begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix}$.



Cluster picture

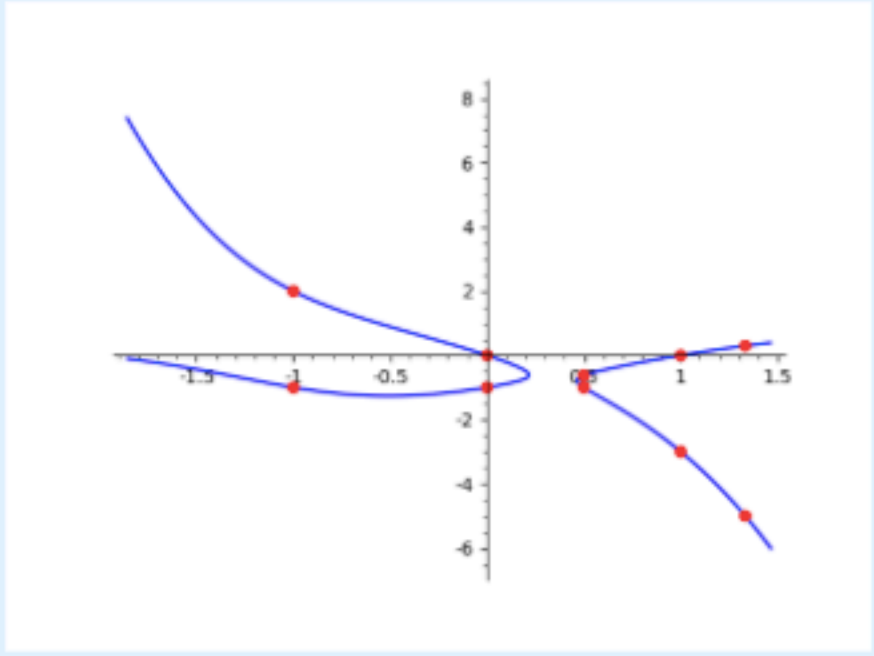


Berkovich space decomposition

$$h_{Z,3}(x,y) = \begin{cases} -\frac{1}{4}\log_p(3) & \text{if } x \equiv -1 \bmod 3, \\ +\frac{1}{4}\log_p(3) & \text{if } x \equiv +1 \bmod 3, \\ 0 & \text{otherwise.} \end{cases}$$

Properties

Label	8649.a.77841.1



Conductor	8649
Discriminant	77841
Mordell-Weil group	$\mathbb{Z} \oplus \mathbb{Z}$
Sato-Tate group	$\mathrm{SU}(2) \times \mathrm{SU}(2)$
$\mathrm{End}(J_{\overline{\mathbb{Q}}}) \otimes \mathbb{R}$	$\mathbb{R} \times \mathbb{R}$
$\mathrm{End}(J_{\overline{\mathbb{Q}}}) \otimes \mathbb{Q}$	RM
$\mathrm{End}(J) \otimes \mathbb{Q}$	RM

Prime	Cluster picture
3	

What is next?

Our implementation handles any hyperelliptic curve C that:

1. is given by an affine model $y^2 = f(x)$, where $f(x)$ is separable and has even degree > 3;
2. does not have bad reduction at 2;
3. has a Jacobian with a nontrivial endomorphism.

Next step: run the quadratic Chabauty method with the heights we compute.