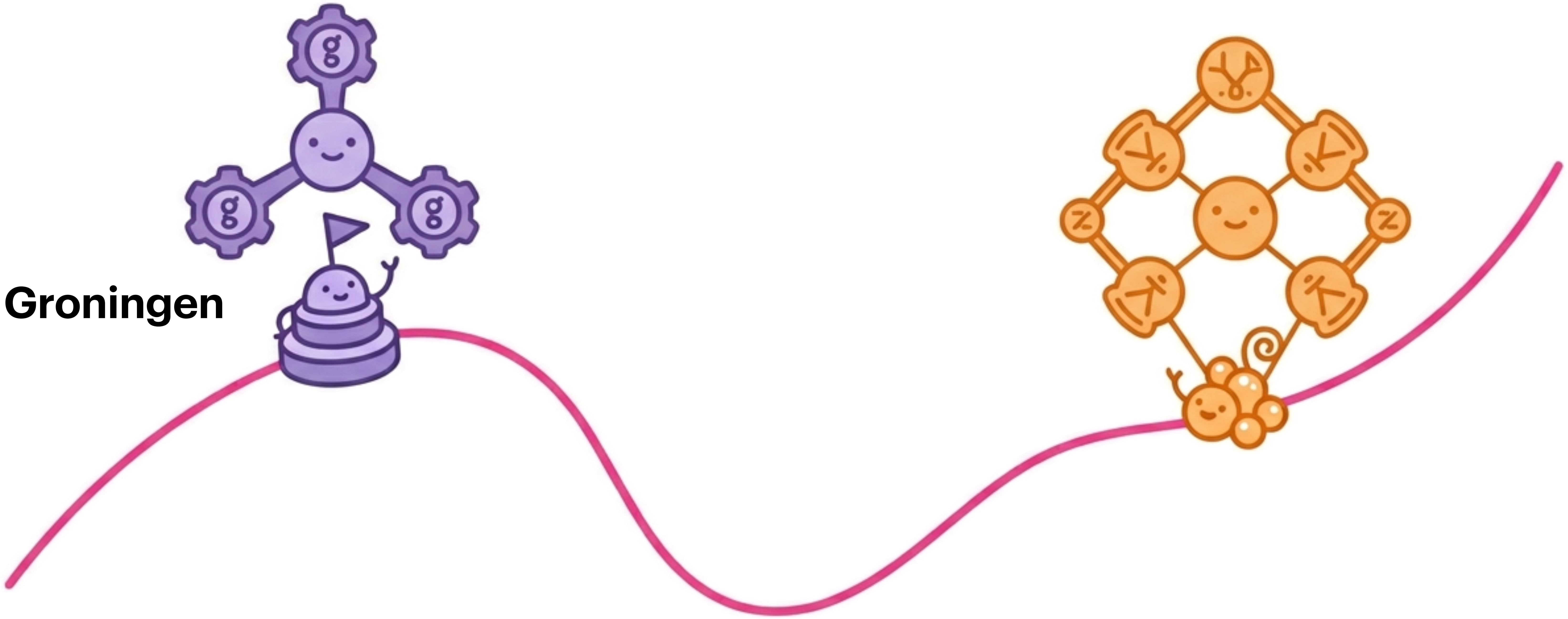
Stacky curves and the local-to-global principle



Juanita Duque-Rosero
Boston University ⟶ University of Groningen

Joint work with Christopher Keyes, Andrew Kobin, Manami Roy, Soumya Sankar, and Yidi Wang.

Can you find nonzero integers x, y, and z such that

$$x^2 + y^2 = 5z^2\,?$$

Can you find nonzero integers x, y, and z such that

$$x^2 + y^2 = 5z^2\ ?$$

$$x = 1, \quad y = 2, \quad z = 1$$

Can you find nonzero integers x, y, and z such that

$$x^2 + y^2 = 5z^2\,?$$

$$x = 1, \quad y = 2, \quad z = 1$$

We can rescale to get infinitely many:

$$x = -3, y = -6, z = -3$$

$$x = 7,\ y = 14,\ z = 7$$

$$x = \lambda,\ y = 2\lambda,\ z = \lambda$$

Can you find nonzero integers x, y, and z such that

$$x^2 + y^2 = 5z^2 \,?$$

Definition. A nonzero solution (x, y, z) is **primitive** if $\gcd(x, y, z) = 1$.

Can you find a primitive integer solution to

$x^2 + y^2 = 3z^2$?

Can you find a primitive integer solution to

$$x^2 + y^2 = 3z^2\ ?$$

NO: If such a solution existed, then there would be a solution to

$$x^2 + y^2 \equiv 0 \pmod 3.$$

y \ x	0	1	2
0	0	1	1
1	1	2	2
2	1	2	2

But then x and y are divisible by 3, which implies that z is too.

Quadratic Forms

Definition. A polynomial $f(x_1, \ldots, x_n)$ is a **quadratic form** is it is homogeneous of degree 2.

Quadratic Forms

Definition. A polynomial $f(x_1, \ldots, x_n)$ is a **quadratic form** is it is homogeneous of degree 2.

Examples. $xy, \quad x^2 + y^2 - 3z^2, \quad 2x_1^2 + x_1x_2 - x_2x_3, \quad x^2 + y^2 + z^2 + w^2.$

Quadratic Forms

Definition. A polynomial $f(x_1, \ldots, x_n)$ is a **quadratic form** is it is homogeneous of degree 2.

Examples. $xy, \quad x^2 + y^2 - 3z^2, \quad 2x_1^2 + x_1x_2 - x_2x_3, \quad x^2 + y^2 + z^2 + w^2.$

Theorem [Hasse–Minkowski]. Let $f(x_1, \ldots, x_n)$ be a non-degenerate quadratic form, where the polynomial f has integer coefficients. Then $f(x_1, \ldots, x_n) = 0$ has a non-trivial solution over $\mathbb{Q}$ if and only if it has a nontrivial solution in $\mathbb{Q}_p$ for all primes p and one over $\mathbb{R}$.

Is there a primitive integral solution for

$$3x^3 + 4y^3 + 5z^3 = 0\ ?$$

Is there a primitive integral solution for

$$3x^3 + 4y^3 + 5z^3 = 0\ ?$$

Is there a primitive integral solution for

$$3x^3 + 4y^3 + 5z^3 = 0\ ?$$

$$3 \cdot 1^3 + 4 \cdot 0^3 + 5\left(\sqrt[3]{-\frac{3}{5}}\right)^3 = 0$$

Is there a primitive integral solution for

$$3x^3 + 4y^3 + 5z^3 = 0 \,?$$

$$3 \cdot 1^3 + 4 \cdot 0^3 + 5\left(\sqrt[3]{-\frac{3}{5}}\right)^3 = 0$$

Is there a primitive integral solution for

$$3x^3 + 4y^3 + 5z^3 = 0\ ?$$

$$3 \cdot 1^3 + 4 \cdot 0^3 + 5\left(\sqrt[3]{-\frac{3}{5}}\right)^3 = 0$$

$$3 \cdot 1^2 + 5 \cdot 1^3 \equiv 1^2 + 1^3 \equiv 0 \pmod 2$$

Is there a primitive integral solution for

$$3x^3 + 4y^3 + 5z^3 = 0\,?$$

$$3 \cdot 1^3 + 4 \cdot 0^3 + 5\left(\sqrt[3]{-\frac{3}{5}}\right)^3 = 0$$

$$3 \cdot 1^2 + 5 \cdot 1^3 \equiv 1^2 + 1^3 \equiv 0 \pmod 2$$

$$4 \cdot 2^3 + 5 \cdot 2^3 \equiv 2^3 + 2 \cdot 2^3 \equiv 0 \pmod 3$$

Is there a primitive integral solution for

$$3x^3 + 4y^3 + 5z^3 = 0\,?$$

$$3 \cdot 1^3 + 4 \cdot 0^3 + 5\left(\sqrt[3]{-\frac{3}{5}}\right)^3 = 0$$

$$3 \cdot 1^2 + 5 \cdot 1^3 \equiv 1^2 + 1^3 \equiv 0 \pmod 2$$

$$4 \cdot 2^3 + 5 \cdot 2^3 \equiv 2^3 + 2 \cdot 2^3 \equiv 0 \pmod 3$$

$$3 \cdot 2^3 + 4 \cdot 4^3 \equiv 3 \cdot 2^3 - (-1)^3 \equiv 0 \pmod 5$$

Is there a primitive integral solution for

$$3x^3 + 4y^3 + 5z^3 = 0\,?$$

$$3 \cdot 1^3 + 4 \cdot 0^3 + 5\left(\sqrt[3]{-\frac{3}{5}}\right)^3 = 0$$

$$3 \cdot 1^2 + 5 \cdot 1^3 \equiv 1^2 + 1^3 \equiv 0 \pmod 2$$

$$4 \cdot 2^3 + 5 \cdot 2^3 \equiv 2^3 + 2 \cdot 2^3 \equiv 0 \pmod 3$$

$$3 \cdot 2^3 + 4 \cdot 4^3 \equiv 3 \cdot 2^3 - (-1)^3 \equiv 0 \pmod 5$$

Exercise. Prove that for any $p \geq 7$, there is always a nonzero solution.

Is there a primitive integral solution for

$$3x^3 + 4y^3 + 5z^3 = 0\,?$$

Is there a primitive integral solution for

$$3x^3 + 4y^3 + 5z^3 = 0\,?$$

Claim. $3x^3 + 4y^3 + 5z^3 = 0$ has a nonzero solution modulo p for all primes and one over $\mathbb{R}$.

Is there a primitive integral solution for

$$3x^3 + 4y^3 + 5z^3 = 0\ ?$$

Claim. $3x^3 + 4y^3 + 5z^3 = 0$ has a nonzero solution modulo p for all primes and one over $\mathbb{R}$.

Claim. $3x^3 + 4y^3 + 5z^3 = 0$ has no primitive integral solution.

Is there a primitive integral solution for
$$3x^3 + 4y^3 + 5z^3 = 0\,?$$

Claim. $3x^3 + 4y^3 + 5z^3 = 0$ has a nonzero solution modulo p for all primes and one over $\mathbb{R}$.

Claim. $3x^3 + 4y^3 + 5z^3 = 0$ has no primitive integral solution.

Proof idea. Look at cubes in $(\mathbb{Z}/p\mathbb{Z})^\times$.

The difference:

$$x^2 + y^2 - 5z^2$$

$$x^2 + y^2 - 3z^2$$

Genus 0 curves

$$3x^3 + 4y^3 + 5z^3$$

Genus 1 curve

The difference:

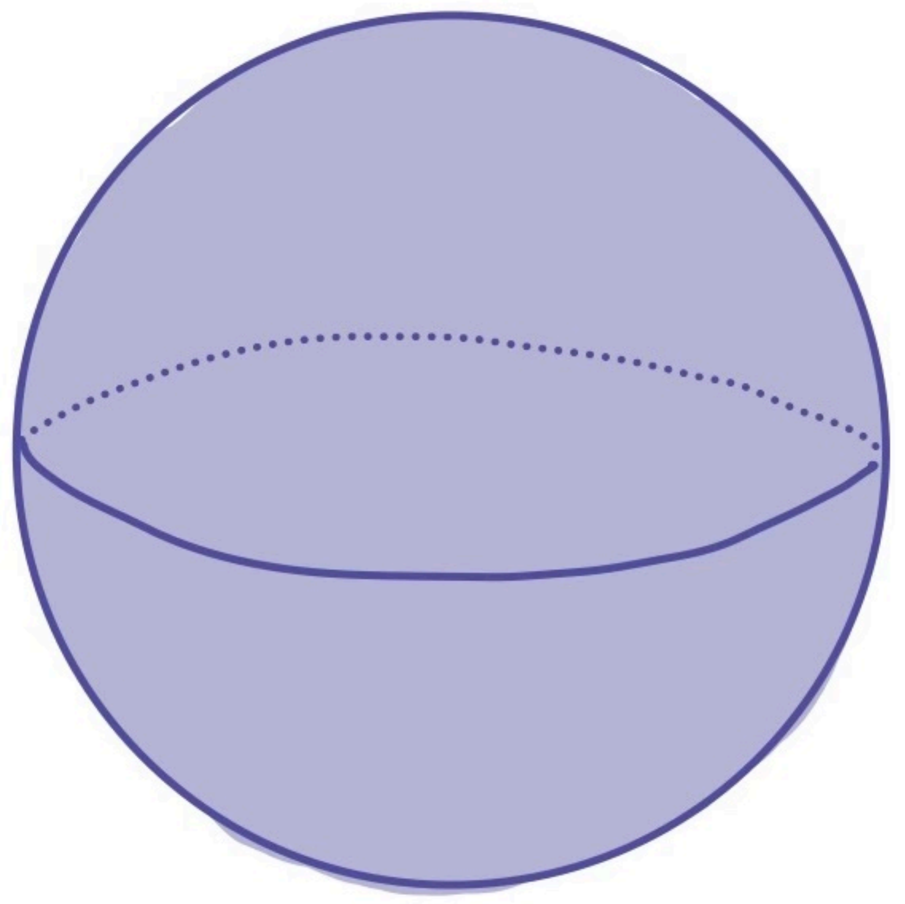
$$x^2 + y^2 - 5z^2$$

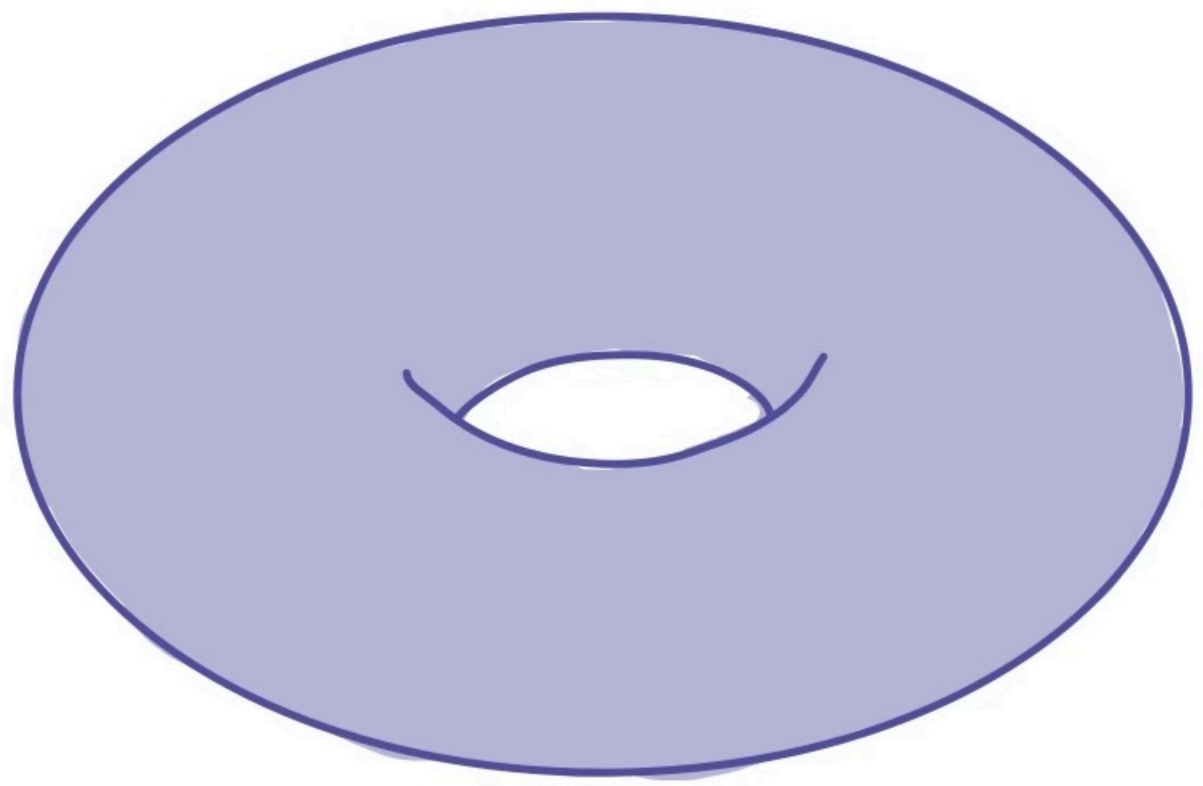
$$x^2 + y^2 - 3z^2$$

$$3x^3 + 4y^3 + 5z^3$$

Genus 0 curves

Genus 1 curve





Curves

Curves

Let $X : f(x, y, z) = 0$ be a curve defined over $\mathbb{Q}$.

A solution $(x_0, y_0, z_0) \in \mathbb{Q}^3$ to the equation $f(x, y, z) = 0$ is called a $\mathbb{Q}$-point on X.

The set of $\mathbb{Q}$-points on X is denoted by $X(\mathbb{Q})$.

Curves

Let $X : f(x, y, z) = 0$ be a curve defined over $\mathbb{Q}$.

A solution $(x_0, y_0, z_0) \in \mathbb{Q}^3$ to the equation $f(x, y, z) = 0$ is called a $\mathbb{Q}$-point on X.

The set of $\mathbb{Q}$-points on X is denoted by $X(\mathbb{Q})$.

Definition. A curve X (over $\mathbb{Q}$) satisfies the **local-to-global principle** if

$$X(\mathbb{Q}_p) \neq \varnothing \text{ for all primes } p \text{ and } X(\mathbb{R}) \neq \varnothing \quad \Longrightarrow \quad X(\mathbb{Q}) \neq \varnothing.$$

Curves

Let $X\colon f(x,y,z) = 0$ be a curve defined over $\mathbb{Q}$.

A solution $(x_0, y_0, z_0) \in \mathbb{Q}^3$ to the equation $f(x,y,z) = 0$ is called a $\mathbb{Q}$-point on X.

The set of $\mathbb{Q}$-points on X is denoted by $X(\mathbb{Q})$.

Definition. A curve X (over $\mathbb{Q}$) satisfies the **local-to-global principle** if

$$X(\mathbb{Q}_p) \neq \varnothing \text{ for all primes } p \text{ and } X(\mathbb{R}) \neq \varnothing \quad \Longrightarrow \quad X(\mathbb{Q}) \neq \varnothing.$$

Note: We can also consider integral points (defined over $\mathbb{Z}$).

Definition. A curve X (over $\mathbb{Q}$) satisfies the **local-to-global principle** if

$$X(\mathbb{Q}) \neq \varnothing \quad \Longrightarrow \quad X(\mathbb{Q}_p) \neq \varnothing \text{ for all primes } p \text{ and } X(\mathbb{R}) \neq \varnothing.$$

To check if a curve satisfies the local-to-global principle over $\mathbb{Q}$ (similar for $\mathbb{Z}$), we need to check:

(1) Are there *local* $\mathbb{Q}_p$- points for all primes p and $\mathbb{R}$-points?

(2) Are there *global* $\mathbb{Q}$-points?

Both answers should be the same.

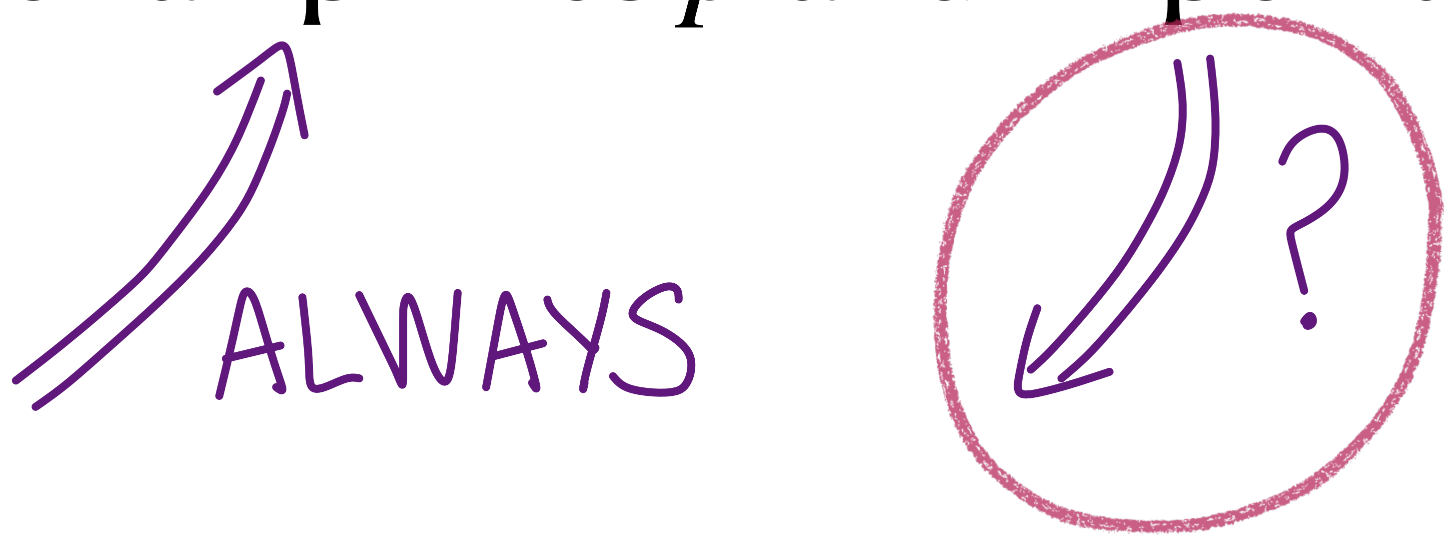
Definition. A curve X (over $\mathbb{Q}$) satisfies the **local-to-global principle** if

$$X(\mathbb{Q}) \neq \varnothing \quad \Longrightarrow \quad X(\mathbb{Q}_p) \neq \varnothing \text{ for all primes } p \text{ and } X(\mathbb{R}) \neq \varnothing.$$

To check if a curve satisfies the local-to-global principle over $\mathbb{Q}$ (similar for $\mathbb{Z}$), we need to check:

(1) Are there *local* $\mathbb{Q}_p$- points for all primes p and $\mathbb{R}$-points?

(2) Are there *global* $\mathbb{Q}$-points?



Both answers should be the same.

Local-to-global principle

Local-to-global principle

Theorem [Hasse–Minkowski]. Let X be a smooth projective geometrically integral curve of genus 0 over $\mathbb{Q}$. Then X satisfies the local-to-global principle over $\mathbb{Q}$.

Local-to-global principle

nice

Theorem [Hasse–Minkowski]. Let X be a ~~smooth projective geometrically integral~~ curve of genus 0 over $\mathbb{Q}$. Then X satisfies the local-to-global principle over $\mathbb{Q}$.

Local-to-global principle

nice

Theorem [Hasse–Minkowski]. Let X be a ~~smooth projective geometrically integral~~ curve of genus 0 over $\mathbb{Q}$. Then X satisfies the local-to-global principle over $\mathbb{Q}$.

Theorem [Bhargava '13]. A positive proportion of genus 1 curves in the weighted projective space given by $z^2 = f(x, y)$, where $f(x, y)$ is an integral binary quadratic form, violate the local-to-global principle over $\mathbb{Q}$.

Is there a primitive integral solution for $x^2 + 3y^2 - z^3 = 0$?

Is there a primitive integral solution for $x^2 + 3y^2 - z^3 = 0$?

Note: the equation does not define a curve, but a surface!

Is there a primitive integral solution for $x^2 + 3y^2 - z^3 = 0$?

Note: the equation does not define a curve, but a surface!

Is there a primitive integral solution for $x^2 + 3y^2 - z^3 = 0$?

$$f(x, y, z) := x^2 + 3y^2 - z^3$$

Is there a primitive integral solution for $x^2 + 3y^2 - z^3 = 0$?

$$f(x, y, z) := x^2 + 3y^2 - z^3$$

Solutions:

$\mathrm{Spec}\, \mathbb{Z}[x, y, z]/f(x, y, z)$

Is there a primitive integral solution for $x^2 + 3y^2 - z^3 = 0$?

$$f(x, y, z) := x^2 + 3y^2 - z^3$$

Solutions:

Spec $\mathbb{Z}[x, y, z]/f(x, y, z)$

Primitive solutions:

???

$$f(x, y, z) := x^2 + 3y^2 - z^3$$

$$f(x, y, z) := x^2 + 3y^2 - z^3$$

- Let (x_0, y_0, z_0) be a solution. Then $(\lambda^3 x_0, \lambda^3 y_0, \lambda^2 z_0)$ is also a solution.

$$f(x, y, z) := x^2 + 3y^2 - z^3$$

- Let (x_0, y_0, z_0) be a solution. Then $(\lambda^3 x_0, \lambda^3 y_0, \lambda^2 z_0)$ is also a solution.
- There is an action of $\mathbb{G}_m$ on

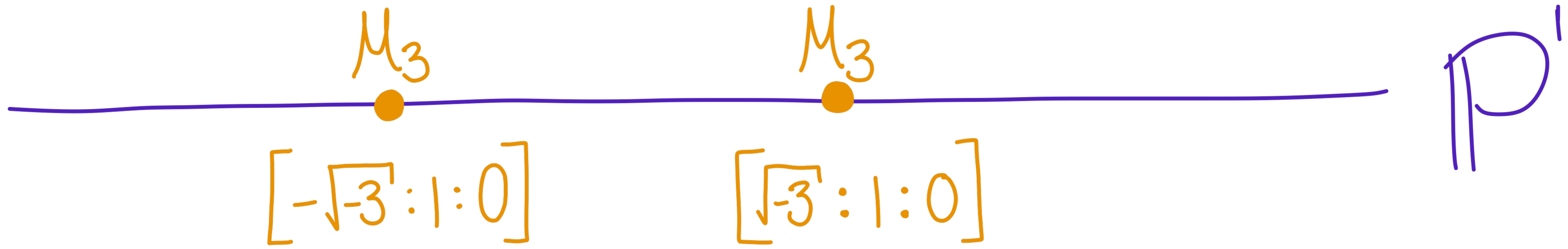
$$S := \operatorname{Spec} \mathbb{Z}[x, y, z]/(f(x, y, z)) \setminus \{x = y = z = 0\} \subseteq \mathbb{A}^3_{\mathbb{Z}} \setminus \{(0,0,0)\}.$$

$$f(x, y, z) := x^2 + 3y^2 - z^3$$

- Let (x_0, y_0, z_0) be a solution. Then $(\lambda^3 x_0, \lambda^3 y_0, \lambda^2 z_0)$ is also a solution.
- There is an action of $\mathbb{G}_m$ on

$$S := \operatorname{Spec} \mathbb{Z}[x, y, z]/(f(x, y, z)) \setminus \{x = y = z = 0\} \subseteq \mathbb{A}^3_{\mathbb{Z}} \setminus \{(0,0,0)\}.$$

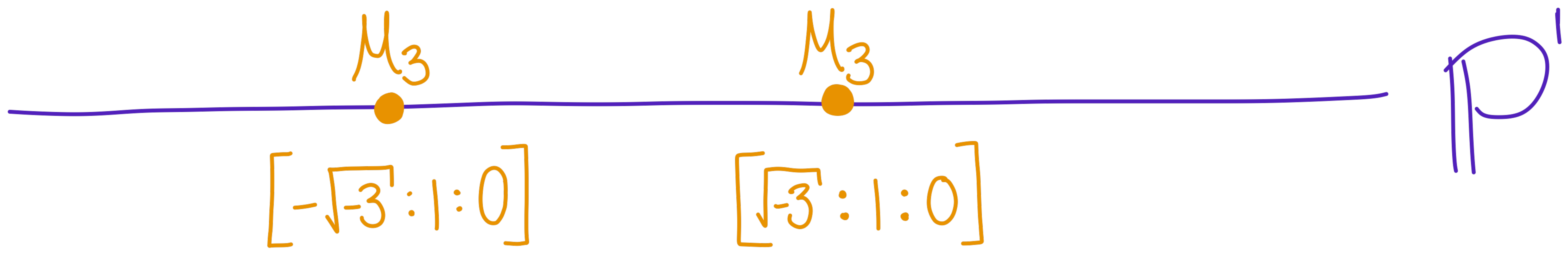
- Consider the *quotient stack* $\mathcal{X} := [S/\mathbb{G}_m]$:

$\mathcal{X}$ 

$$f(x, y, z) := x^2 + 3y^2 - z^3$$

- Let (x_0, y_0, z_0) be a solution. Then $(\lambda^3 x_0, \lambda^3 y_0, \lambda^2 z_0)$ is also a solution.
- Consider the *quotient stack* $\mathscr{X} := [S/\mathbb{G}_m]$:

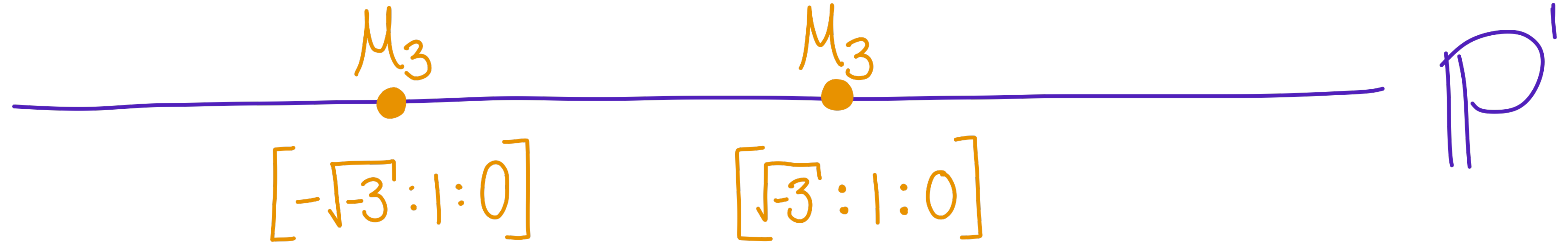
$\mathscr{X}$



$$f(x, y, z) := x^2 + 3y^2 - z^3$$

- Let (x_0, y_0, z_0) be a solution. Then $(\lambda^3 x_0, \lambda^3 y_0, \lambda^2 z_0)$ is also a solution.
- Consider the *quotient stack* $\mathscr{X} := [S/\mathbb{G}_m]$:

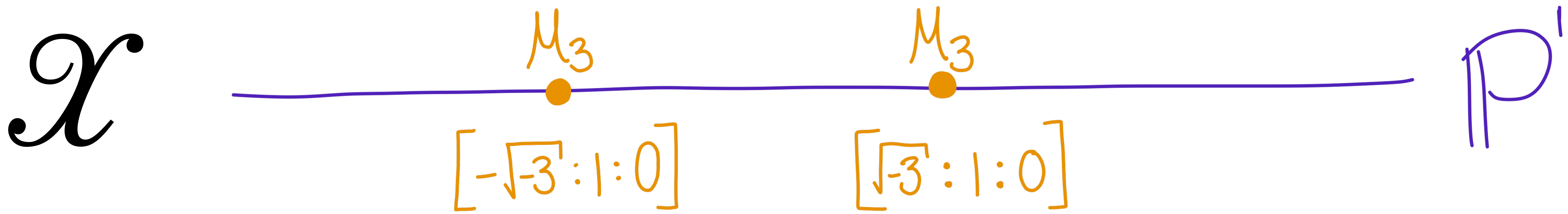
$\mathscr{X}$



Primitive integral solutions to $f(x, y, z) = 0$ $\iff$ Integral points on $\mathscr{X}$: $\mathscr{X}(\mathbb{Z})$

Stacky curves

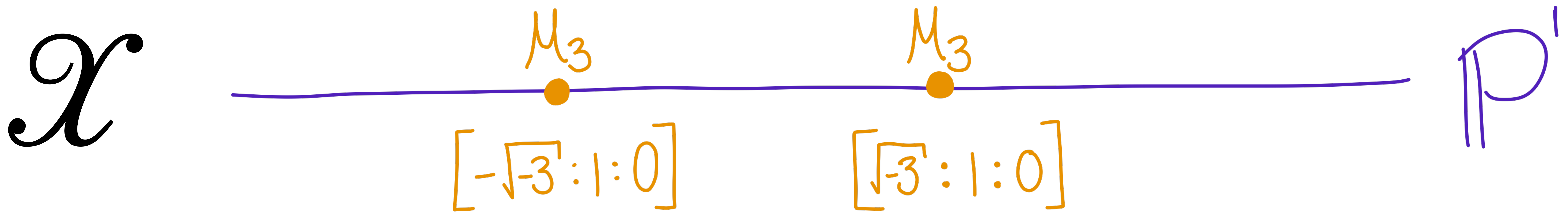
Definition. A stacky curve $\mathscr{X}$ over k is a smooth proper geometrically connected Deligne–Mumford stack of dimension 1 over k that contains a dense open subscheme.



Stacky curves

Definition. A stacky curve $\mathscr{X}$ over k is a smooth proper geometrically connected Deligne–Mumford stack of dimension 1 over k that contains a dense open subscheme.

Think: an algebraic curve (coarse space) with finitely many stacky points x each with stacky stabilizer group G_x.

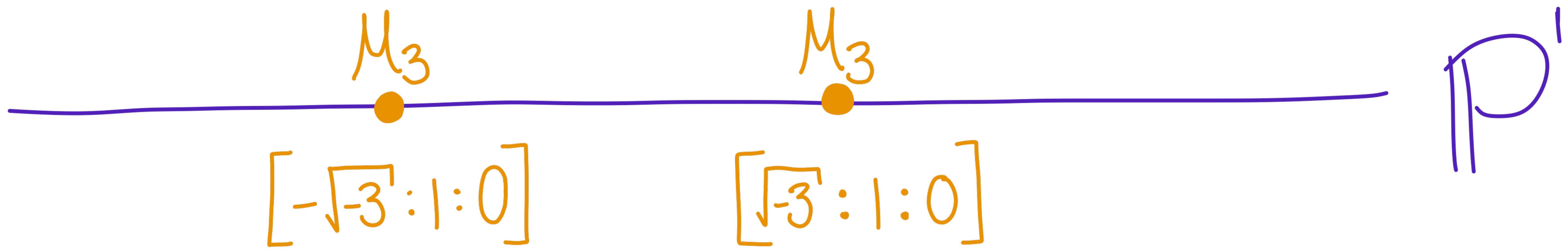


Stacky curves

Definition. A stacky curve $\mathscr{X}$ over k is a smooth proper geometrically connected Deligne–Mumford stack of dimension 1 over k that contains a dense open subscheme.

Think: an algebraic curve (coarse space) with finitely many stacky points x each with stabilizer group G_x.

$\mathscr{X}$



Claim: $\mathscr{X}$ is a stacky curve of genus 2/3.

Local-to-global principle

Local-to-global principle

Theorem [Bhargava & Poonen '21]. Let $\mathscr{X}$ be a stacky curve over $\mathbb{Z}$ of genus less than 1/2. Then $\mathscr{X}$ satisfies the local-to-global principle.

Local-to-global principle

Theorem [Bhargava & Poonen '21]. Let $\mathscr{X}$ be a stacky curve over $\mathbb{Z}$ of genus less than 1/2. Then $\mathscr{X}$ satisfies the local-to-global principle.

Theorem [Bhargava & Poonen '21]. Let

$$S := \operatorname{Spec} \mathbb{Z}[x,y,z]/\left(z^2 - (3x^2 + xy + 850y^2)\right) \smallsetminus \{x = y = z = 0\} \subseteq \mathbb{A}^3_{\mathbb{Z}} \smallsetminus \{(0,0,0)\}$$

Consider the action of μ_2 as $[x : y : z] \mapsto [x : y : \zeta z]$ on $\mathscr{X} := [S/\mathbb{G}_m]$. Let $\mathscr{Y}$ be the quotient stack $[\mathscr{X}/\mu_2]$. Then

(a) the genus of $\mathscr{Y}$ is 1/2;

(b) $\mathscr{Y}(\mathbb{Z}_p) \neq \varnothing$ for every rational prime p and $\mathscr{Y}(\mathbb{R}) \neq \varnothing$;

(c) $\mathscr{Y}(\mathbb{Z}) \neq \varnothing$.

An infinite family

An infinite family

Consider the polynomials $f_{B,C,n}(x, y, z) := x^2 + By^2 - Cz^n$ for n odd.

An infinite family

Consider the polynomials $f_{B,C,n}(x, y, z) := x^2 + By^2 - Cz^n$ for n odd.

An infinite family

Consider the polynomials $f_{B,C,n}(x, y, z) := x^2 + By^2 - Cz^n$ for n odd.

These define *generalized Fermat curves*.

Studied by [Darmon & Granville '94].

An infinite family

Consider the polynomials $f_{B,C,n}(x,y,z) := x^2 + By^2 - Cz^n$ for n odd.

These define *generalized Fermat curves.*

Studied by [Darmon & Granville '94].

$$S := \operatorname{Spec} \mathbb{Z}[x,y,z]/(f_{B,C,n}(x,y,z)) \smallsetminus \{x=y=z=0\} \subseteq \mathbb{A}^3_{\mathbb{Z}} \smallsetminus \{0\}$$

$$\mathscr{X}_{B,C,n} := [S_{B,C,n}/\mathbb{G}_m] \subset [\mathbb{A}^3 \smallsetminus \{0\}/\mathbb{G}_m] = \mathbb{P}(n,n,2)$$

An infinite family

Consider the polynomials $f_{B,C,n}(x, y, z) := x^2 + By^2 - Cz^n$ for n odd.

These define *generalized Fermat curves.*

Studied by [Darmon & Granville '94].

$$S := \operatorname{Spec} \mathbb{Z}[x, y, z]/(f_{B,C,n}(x, y, z)) \setminus \{x = y = z = 0\} \subseteq \mathbb{A}^3_{\mathbb{Z}} \setminus \{0\}$$

$$\mathscr{X}_{B,C,n} := [S_{B,C,n}/\mathbb{G}_m] \subset [\mathbb{A}^3 \setminus \{0\}/\mathbb{G}_m] = \mathbb{P}(n, n, 2)$$

By an analogous to Riemann-Hurwitz, we get that the genus of $\mathscr{X}_{B,C,n}$ is $\dfrac{n-1}{n}$.

An infinite family

Consider the polynomials $f_{B,C,n}(x, y, z) := x^2 + By^2 - Cz^n$ for n odd.

These define *generalized Fermat curves.*

Studied by [Darmon & Granville '94].

$$\mathcal{X}_{B,C,n} : x^2 + By^2 = Cz^n \subset \mathbb{P}(n, n, 2)$$

By an analogous to Riemann-Hurwitz, we get that the genus of $\mathcal{X}_{B,C,n}$ is $\dfrac{n-1}{n}$.

Checking for local-to-global

$$\mathcal{X}_{B,C,n} : x^2 + By^2 = Cz^n \subset \mathbb{P}(n, n, 2)$$

We need to answer:

1. $\mathcal{X}_{B,C,n}(\mathbb{Z}_p) \neq \varnothing$ for all primes p and $\mathcal{X}_{B,C,n}(\mathbb{R}) \neq \varnothing$?
2. $\mathcal{X}_{B,C,n}(\mathbb{Z}) \neq \varnothing$?

1. Are there local $\mathbb{Z}_p$- points and $\mathbb{R}$-points?

$$\mathscr{X}_{B,C,n} : x^2 + By^2 = Cz^n \subset \mathbb{P}(n, n, 2)$$

1. Are there local $\mathbb{Z}_p$- points and $\mathbb{R}$-points?

$$\mathscr{X}_{B,C,n} : x^2 + By^2 = Cz^n \subset \mathbb{P}(n, n, 2)$$

This comes down to Legendre symbols $\left(\frac{-B_0}{p}\right)$ for $p \mid C$, where $B = f^2 B_0$.

1. Are there local $\mathbb{Z}_p$- points and $\mathbb{R}$-points?

$$\mathcal{X}_{B,C,n} : x^2 + By^2 = Cz^n \subset \mathbb{P}(n,n,2)$$

This comes down to Legendre symbols $\left(\frac{-B_0}{p}\right)$ for $p \mid C$, where $B = f^2 B_0$.

Example [$x^2 + 29y^2 = 3z^3$].

The ideal 3 is split in $\mathbb{Q}\left(\sqrt{-29}\right)$. Thus there exist x and y such that

$$x^2 + 29y^2 \equiv 0 \pmod{3}.$$

Conclude $\mathcal{X}_{29,3,3}(\mathbb{Z}_3) \neq \emptyset$.

2. Are there global $\mathbb{Z}$- points?

$$\mathscr{X}_{B,C,n} : x^2 + By^2 = Cz^n \subset \mathbb{P}(n, n, 2)$$

2. Are there global $\mathbb{Z}$-points?

$$\mathcal{X}_{B,C,n} : x^2 + By^2 = Cz^n \subset \mathbb{P}(n,n,2)$$

Theorem [DKKRSW '25]. Write $B = f^2 B_0$ and let $K = \mathbb{Q}\left(\sqrt{-B_0}\right)$ with ring of integers $\mathcal{O}_K$. Assume C and f are coprime. Then there exists a localization R_K of $\mathcal{O}_K$ such that TFAE:

2. Are there global $\mathbb{Z}$- points?

$$\mathscr{X}_{B,C,n} : x^2 + By^2 = Cz^n \subset \mathbb{P}(n, n, 2)$$

Theorem [DKKRSW '25]. Write $B = f^2 B_0$ and let $K = \mathbb{Q}\left(\sqrt{-B_0}\right)$ with ring of integers $\mathcal{O}_K$. Assume C and f are coprime. Then there exists a localization R_K of $\mathcal{O}_K$ such that TFAE:

(i) $\mathscr{X}_{B,C,n}(\mathbb{Z}; \gcd(x, fy) = 1) \neq \emptyset$;

2. Are there global $\mathbb{Z}$-points?

$$\mathcal{X}_{B,C,n}: x^2 + By^2 = Cz^n \subset \mathbb{P}(n, n, 2)$$

Theorem [DKKRSW '25]. Write $B = f^2B_0$ and let $K = \mathbb{Q}\left(\sqrt{-B_0}\right)$ with ring of integers $\mathcal{O}_K$. Assume C and f are coprime. Then there exists a localization R_K of $\mathcal{O}_K$ such that TFAE:

(i) $\mathcal{X}_{B,C,n}(\mathbb{Z}; \gcd(x, fy) = 1) \neq \emptyset$;

(ii) there exists admissible $d \in R_K^\times/(R_K^\times)^n$;

2. Are there global $\mathbb{Z}$- points?

$$\mathscr{X}_{B,C,n} : x^2 + By^2 = Cz^n \subset \mathbb{P}(n, n, 2)$$

Theorem [DKKRSW '25]. Write $B = f^2 B_0$ and let $K = \mathbb{Q}\left(\sqrt{-B_0}\right)$ with ring of integers $\mathcal{O}_K$. Assume C and f are coprime. Then there exists a localization R_K of $\mathcal{O}_K$ such that TFAE:

(i) $\mathscr{X}_{B,C,n}(\mathbb{Z}; \gcd(x, fy) = 1) \neq \emptyset$;

(ii) there exists admissible $d \in R_K^\times/(R_K^\times)^n$;

(iii) $C\mathcal{O}_K = \mathfrak{j}\bar{\mathfrak{j}}\mathfrak{r}^2\mathfrak{i}^2$ with $\mathfrak{j}$, $\mathfrak{r}$, $\mathfrak{i}$ supported on split, ramified, and inert primes and $\left[\mathfrak{j} \cap \mathbb{Z}[f\sqrt{-B_0}]\right] \in n\mathrm{Cl}\left(\mathbb{Z}[f\sqrt{-B_0}]\right)$.

2. Are there global $\mathbb{Z}$- points?

$$\mathcal{X}_{B,C,n} : x^2 + By^2 = Cz^n \subset \mathbb{P}(n, n, 2)$$

2. Are there global $\mathbb{Z}$- points?

$$\mathscr{X}_{B,C,n}: x^2 + By^2 = Cz^n \subset \mathbb{P}(n,n,2)$$

Example [$x^2 + 29y^2 = 3z^3$].

We have $\mathscr{X}_{29,3,3}(\mathbb{Z}_p) \neq \varnothing$ for all primes p.

Brute force: (9,0,3), (10,10,10), …, but no primitive solutions.

2. Are there global $\mathbb{Z}$- points?

$$\mathscr{X}_{B,C,n} : x^2 + By^2 = Cz^n \subset \mathbb{P}(n, n, 2)$$

Example $[x^2 + 29y^2 = 3z^3]$**.**

2. Are there global $\mathbb{Z}$- points?

$$\mathcal{X}_{B,C,n} : x^2 + By^2 = Cz^n \subset \mathbb{P}(n, n, 2)$$

Example [$x^2 + 29y^2 = 3z^3$].

$$K = \mathbb{Q}\left(\sqrt{-29}\right), \quad R_K = \mathcal{O}_K[1/6], \quad 2\mathcal{O}_K = \mathfrak{p}_2^2, \quad 3\mathcal{O}_K = \mathfrak{p}_3\overline{\mathfrak{p}_3}\,.$$

2. Are there global $\mathbb{Z}$-points?

$$\mathcal{X}_{B,C,n}: x^2 + By^2 = Cz^n \subset \mathbb{P}(n,n,2)$$

Example [$x^2 + 29y^2 = 3z^3$].

$$K = \mathbb{Q}\left(\sqrt{-29}\right), \quad R_K = \mathcal{O}_K[1/6], \quad 2\mathcal{O}_K = \mathfrak{p}_2^2, \quad 3\mathcal{O}_K = \mathfrak{p}_3\overline{\mathfrak{p}_3}.$$

Admissible $d \in R_K^\times/(R_K^\times)^3$ satisfy

$$v_{\mathfrak{p}_3}(d) \equiv \pm 1 \pmod 3, \quad v_{\overline{\mathfrak{p}_3}}(d) \equiv \mp 1 \pmod 3, \quad v_{\mathfrak{p}_2}(d) \equiv 0 \pmod 3.$$

2. Are there global $\mathbb{Z}$- points?

$$\mathscr{X}_{B,C,n} : x^2 + By^2 = Cz^n \subset \mathbb{P}(n, n, 2)$$

Example [$x^2 + 29y^2 = 3z^3$].

$K = \mathbb{Q}\left(\sqrt{-29}\right)$, $R_K = \mathcal{O}_K[1/6]$, $2\mathcal{O}_K = \mathfrak{p}_2^2$, $3\mathcal{O}_K = \mathfrak{p}_3\overline{\mathfrak{p}_3}$.

Admissible $d \in R_K^\times/(R_K^\times)^3$ satisfy

$$v_{\mathfrak{p}_3}(d) \equiv \pm 1 \pmod 3, \quad v_{\overline{\mathfrak{p}_3}}(d) \equiv \mp 1 \pmod 3, \quad v_{\mathfrak{p}_2}(d) \equiv 0 \pmod 3.$$

Computation of $R_K^\times/(R_K^\times)^3$ reveals no such d exist, so $\mathscr{X}_{29,3,3}(\mathbb{Z}) = \varnothing$.

How many curves $\mathscr{X}_{B,C,3}$ satisfy the local-to-global principle?

$$N^{\mathrm{loc}}(T) := \#\{(B,C) \in \mathbb{Z}^2 : |B|, |C| < T, \mathscr{X}_{B,C,3}(\mathbb{Z}_p) \neq \varnothing \text{ for all primes } p\},$$

$$N(T) := \#\{(B,C) \in \mathbb{Z}^2 : |B|, |C| < T, \mathscr{X}_{B,C,3}(\mathbb{Z}) \neq \varnothing\}.$$

How many curves $\mathcal{X}_{B,C,3}$ satisfy the local-to-global principle?

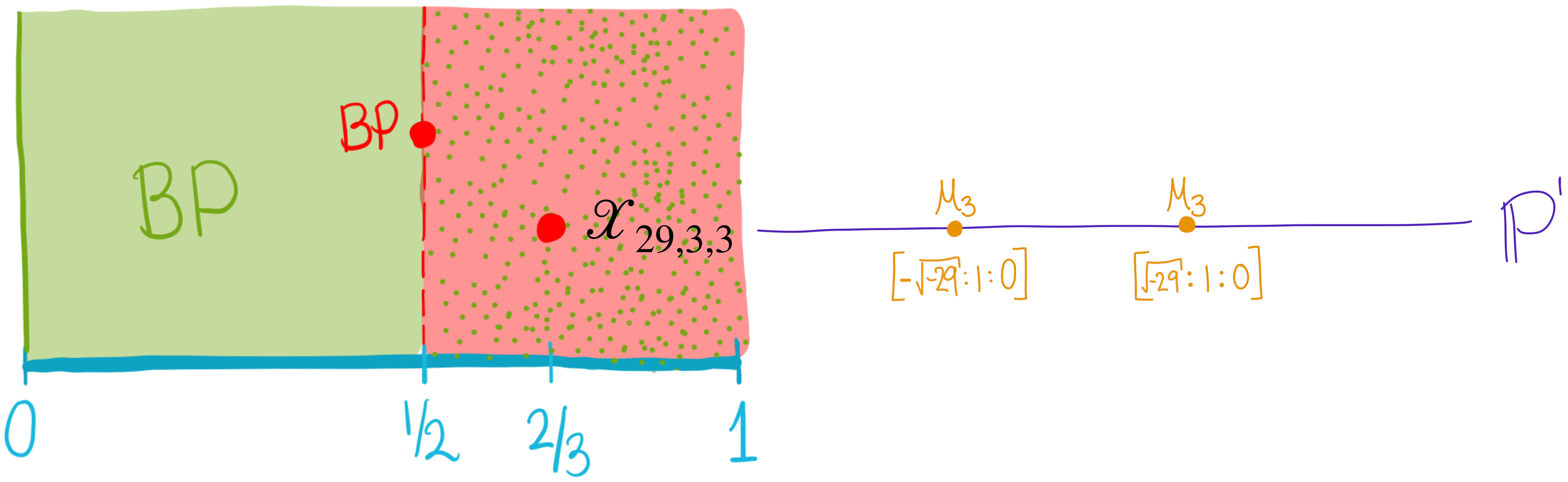
$$N^{\mathrm{loc}}(T) := \#\{(B,C) \in \mathbb{Z}^2 : |B|, |C| < T, \mathcal{X}_{B,C,3}(\mathbb{Z}_p) \neq \varnothing \text{ for all primes } p\},$$

$$N(T) := \#\{(B,C) \in \mathbb{Z}^2 : |B|, |C| < T, \mathcal{X}_{B,C,3}(\mathbb{Z}) \neq \varnothing\}\,.$$

Theorem [DKKRSW '25]. There is a positive proportion of curves $\mathcal{X}_{B,C,3}$ that non-vacuously satisfy the local-to-global principle for integral points. That is,

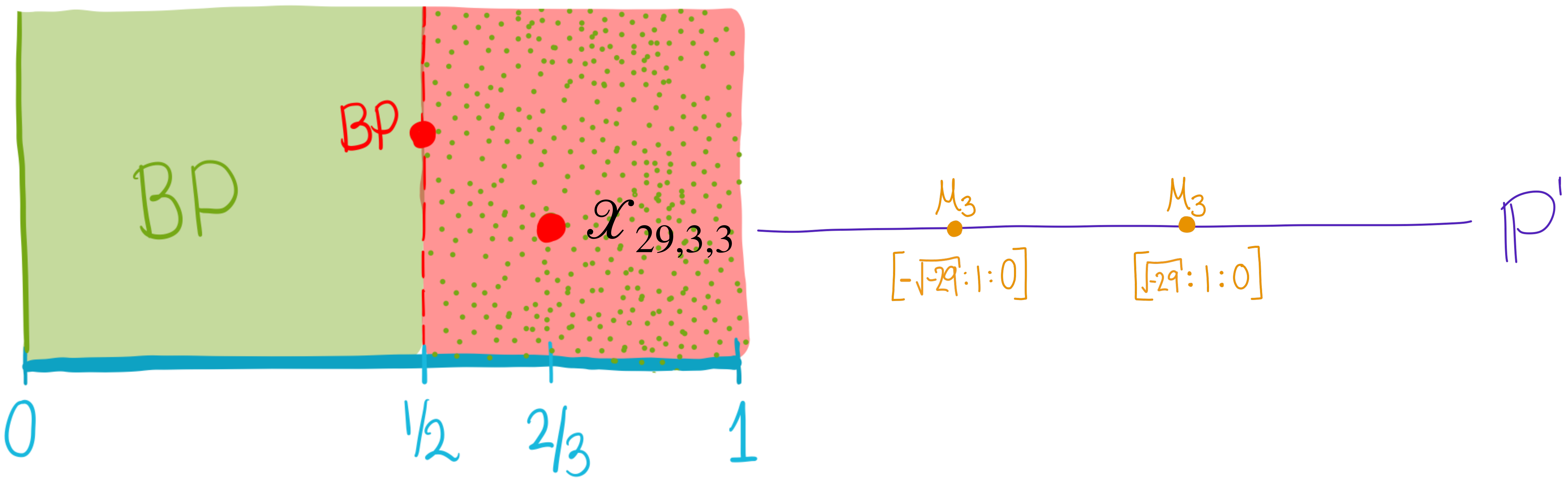
$$\liminf_{T\to\infty} \frac{N(T)}{N^{\mathrm{loc}}(T)} > 0.$$

Thank you!



Theorem [DKKRSW '25]. There is a positive proportion of curves $\mathscr{X}_{B,C,3}$ that non-vacuously satisfy the local-to-global principle for integral points.

Thank you!



Theorem [DKKRSW '25]. There is a positive proportion of curves $\mathscr{X}_{B,C,3}$ that non-vacuously satisfy the local-to-global principle for integral points.

Q. OK, There are solutions... How many are there? Looking forward to Santi's talk tomorrow!