Computing local heights on hyperelliptic curves for quadratic Chabauty

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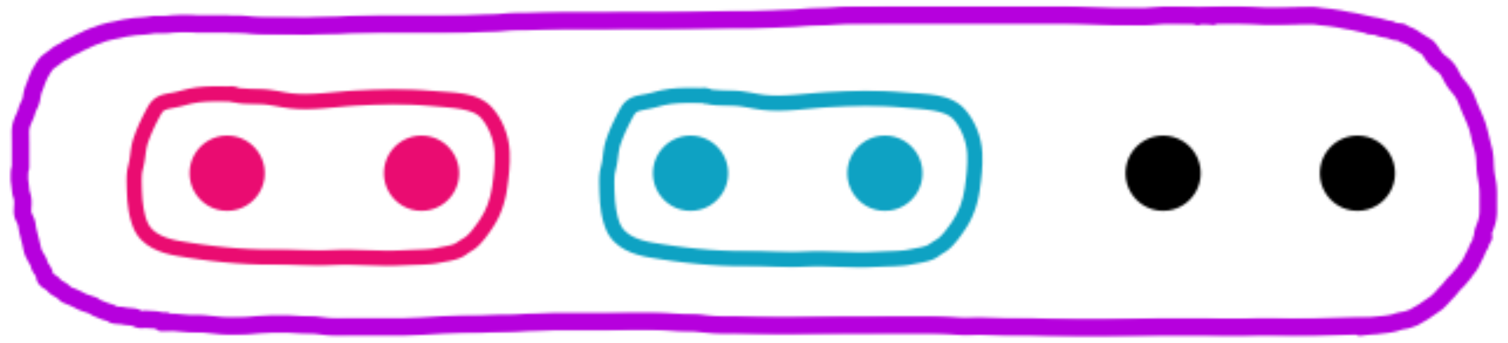
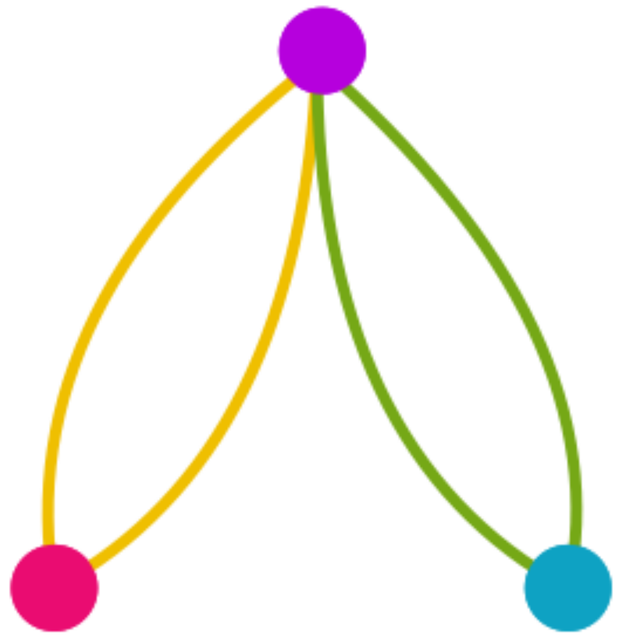
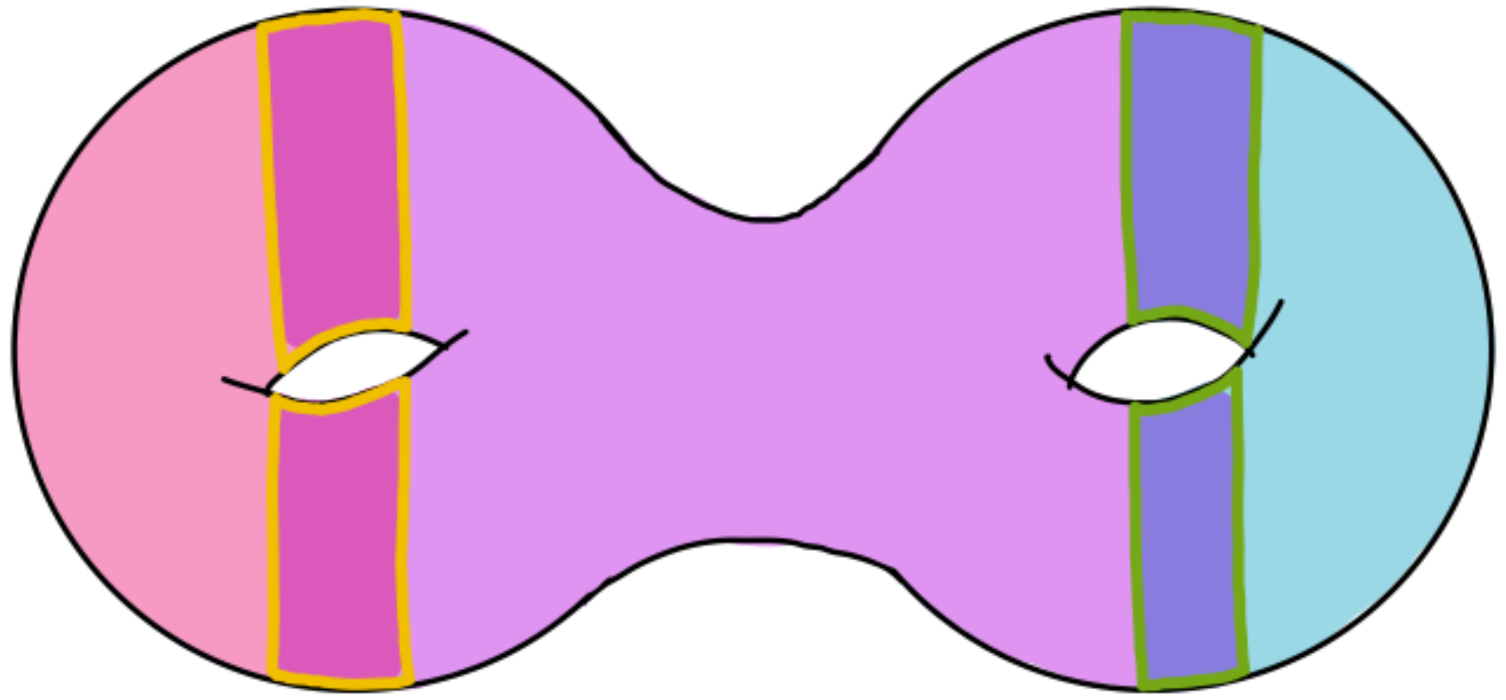
Joint work with Alexander Betts, Sachi Hashimoto, and Pim Spelier

Theorem [Betts-DR-Hashimoto-Spelier, '24]. There are 10 (explicit) rational points on the curve

$$C : y^2 = x^6 + \frac{18}{5}x^4 + \frac{6}{5}x^3 + \frac{9}{5}x^2 + \frac{6}{5}x + \frac{1}{5}.$$

Quadratic Chabauty

Chabauty—Kim's Method [‘05, ‘09]. Goal is to use p-adic methods to determine $C(\mathbb{Q})$.

Balakrishnan—Dogra [‘18, ‘20]. Chabauty—Kim's Program is made explicit for $r = g$ $(+\epsilon)$.

Theorem [Balakrishnan—Dogra, '18]. Suppose that $r = g$ $(+\epsilon)$, and that $Z \subset C \times C$ is a trace 0 correspondence invariant under the Rosati involution. Then there exists a quadratic function

$$Q_Z\colon \operatorname{Lie}\left(J_{\mathbb{Q}_p}\right) \to \mathbb{Q}_p$$

for which $C(\mathbb{Q})$ is contained in the locus inside $C(\mathbb{Q}_p)$ cut out by the equation

$$Q_Z(\log(z)) - h_{Z,p}(z) \in \Omega,$$

where $\Omega = \left\{ \sum_{\ell \neq p} h_{Z,\ell}(x_\ell) \ : \ x_\ell \in C(\mathbb{Q}_\ell) \right\}$.

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Quadratic Chabauty and heights

Key input: Let p be a prime number and $Z \subset C \times C$ be a trace 0 correspondence. There the associated p-**adic** (Coleman—Gross) **height function** $h_Z : C(\mathbb{Q}) \to \mathbb{Q}_p$ can be decomposed as

$$h_Z(Q) = \sum_{\ell} h_{Z,\ell}(Q),$$

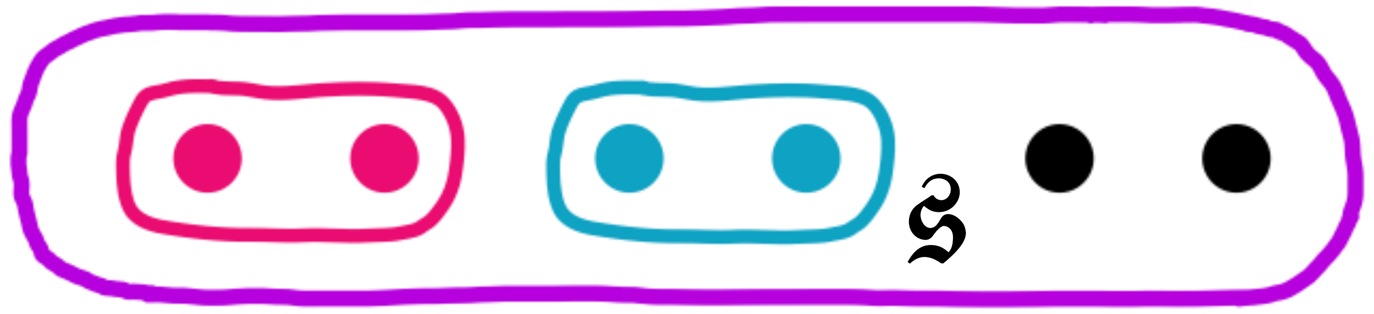
where $h_{Z,\ell} : C(\mathbb{Q}_\ell) \to \mathbb{Q}_p$.

- For $\ell \neq p$ the height function $h_{Z,\ell}$ takes only finitely many values.
- If $\ell \neq p$ is a prime of potential good reduction, then $h_{Z,\ell} = 0$.
- Methods for computing $h_{Z,\ell}$ have depended on constructing a regular semistable model of C over $\mathbb{Z}$, which can be extremely difficult, especially as the genus or conductor of C gets large!

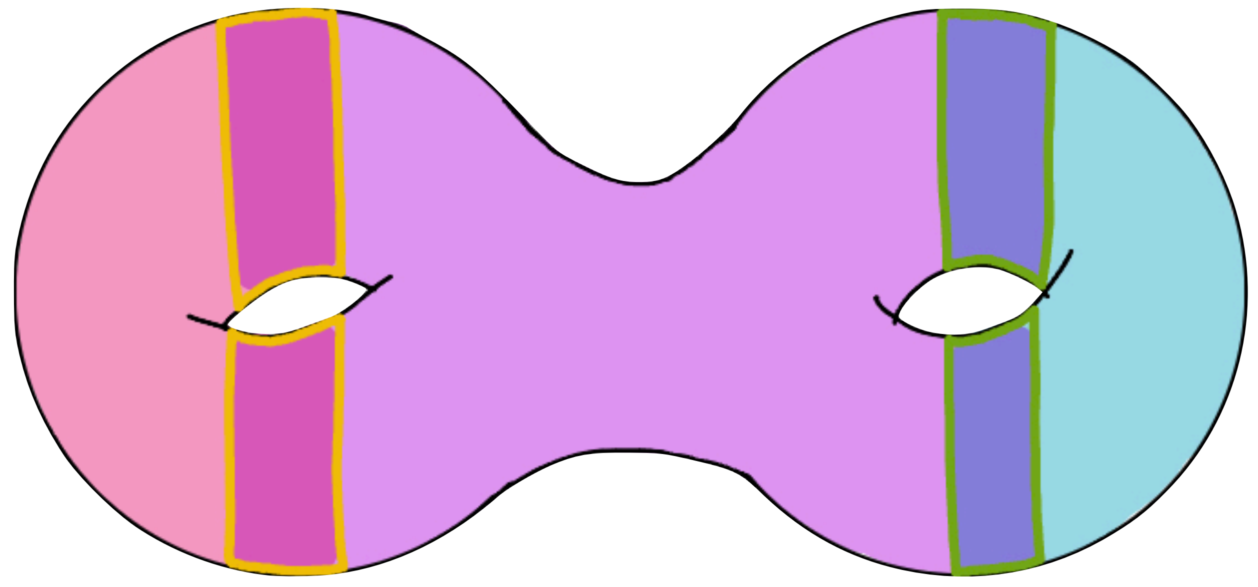
Today's problem: **to compute heights $h_{Z,\ell}$ for $\ell \neq p$ of bad reduction**.

Computing local heights: how?

- **Theorem [Betts–Dogra, '20]**. There is an explicit formula for computing $h_{Z,\ell}$ in terms of the induced action of Z_* on the homology $\mathrm{H}_1(\Gamma, \mathbb{Q})$ of the dual graph Γ of the geometric special fibre.
- This formula uses a semistable model of C. Instead, we construct a semistable covering of C using cluster pictures [Best et al., '22].
- We then prove a version of the Coleman–Iovita isomorphism that relates the cohomology of the reduction graph Γ associated to a semistable model C to the de Rham cohomology of C (where we compute the action of Z).



Cluster picture: roots of $f(x)$.



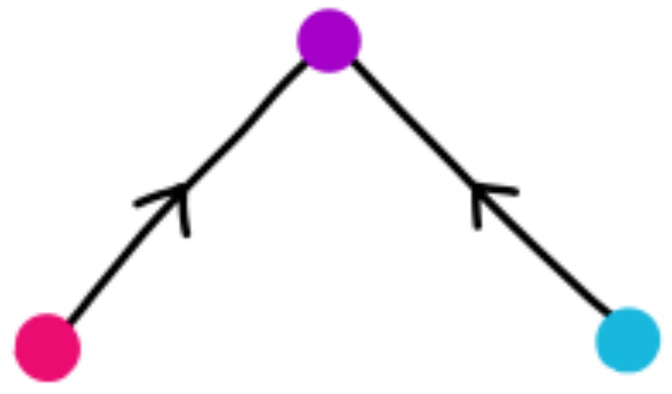
Semistable covering of C^{an}

An example

$$C : y^2 = x^6 + \frac{18}{5}x^4 + \frac{6}{5}x^3 + \frac{9}{5}x^2 + \frac{6}{5}x + \frac{1}{5}$$

- $g = r = 2.$
- The Jacobian of C has real multiplication by $\sqrt{13}$.
- The primes of bad reduction are $\ell = 3, 5$.

$\ell = 3$



$$h_{Z,3}(P) = \begin{cases} 0 & \text{if } P = \infty; \\ 0 & \text{if } v_3(x) < 0; \\ 8/3 & \text{if } v_3(x-1) \geq 1; \\ 0 & \text{otherwise}. \end{cases}$$

An example

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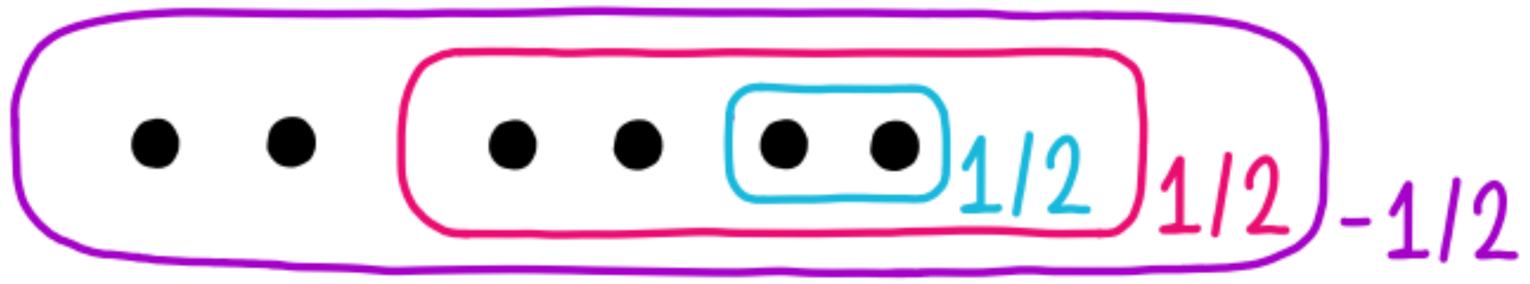

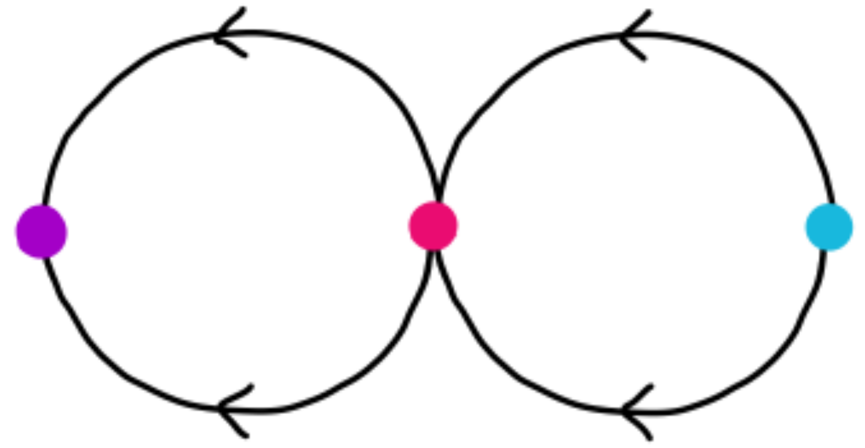

$$h_{Z,5}(P) = \begin{cases} -3 & \text{if } P = \infty; \\ -3 & \text{if } v_5(x) < 0; \\ 3 & \text{if } v_5(x-2) \geq 1; \\ 0 & \text{otherwise}. \end{cases}$$

An example (ctd.)

We can then apply the quadratic Chabauty method, together with a Mordell-Weil sieve [Balakrishnan, Dogra, Müller, Tuitman, and Vonk; Stoll]

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Available at your nearest arXiv starting today!

Thank you.