Triangular modular curves

Juanita Duque-Rosero

Joint work with John Voight

DARTMOUTH

Research supported by: Simons Collaborations in the Mathematics and the Physical Sciences Award, Arithmetic geometry, number theory, and computation

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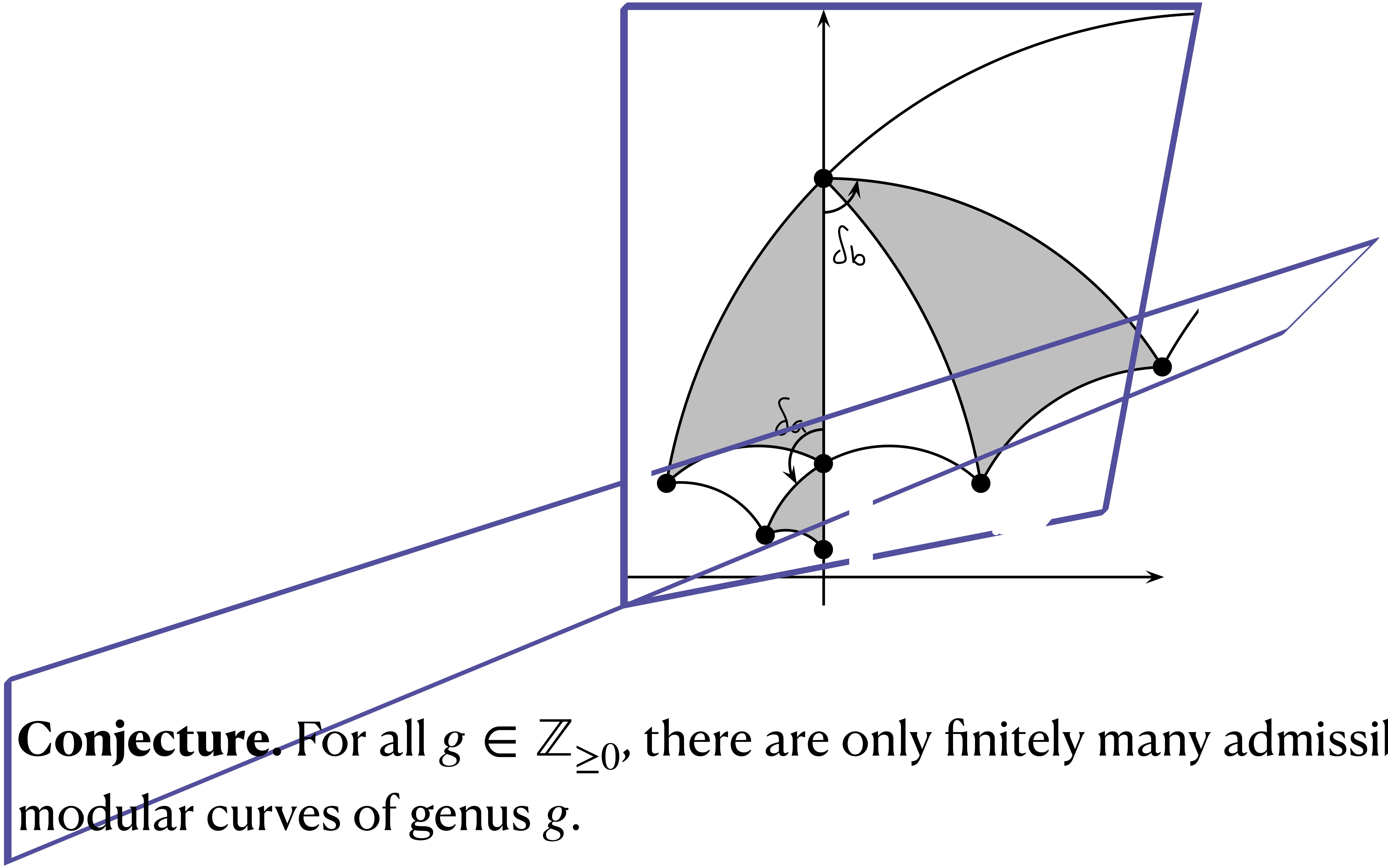
Thank you!



Conjecture. For all $g \in \mathbb{Z}_{\geq 0}$, there are only finitely many admissible triangular modular curves of genus g.

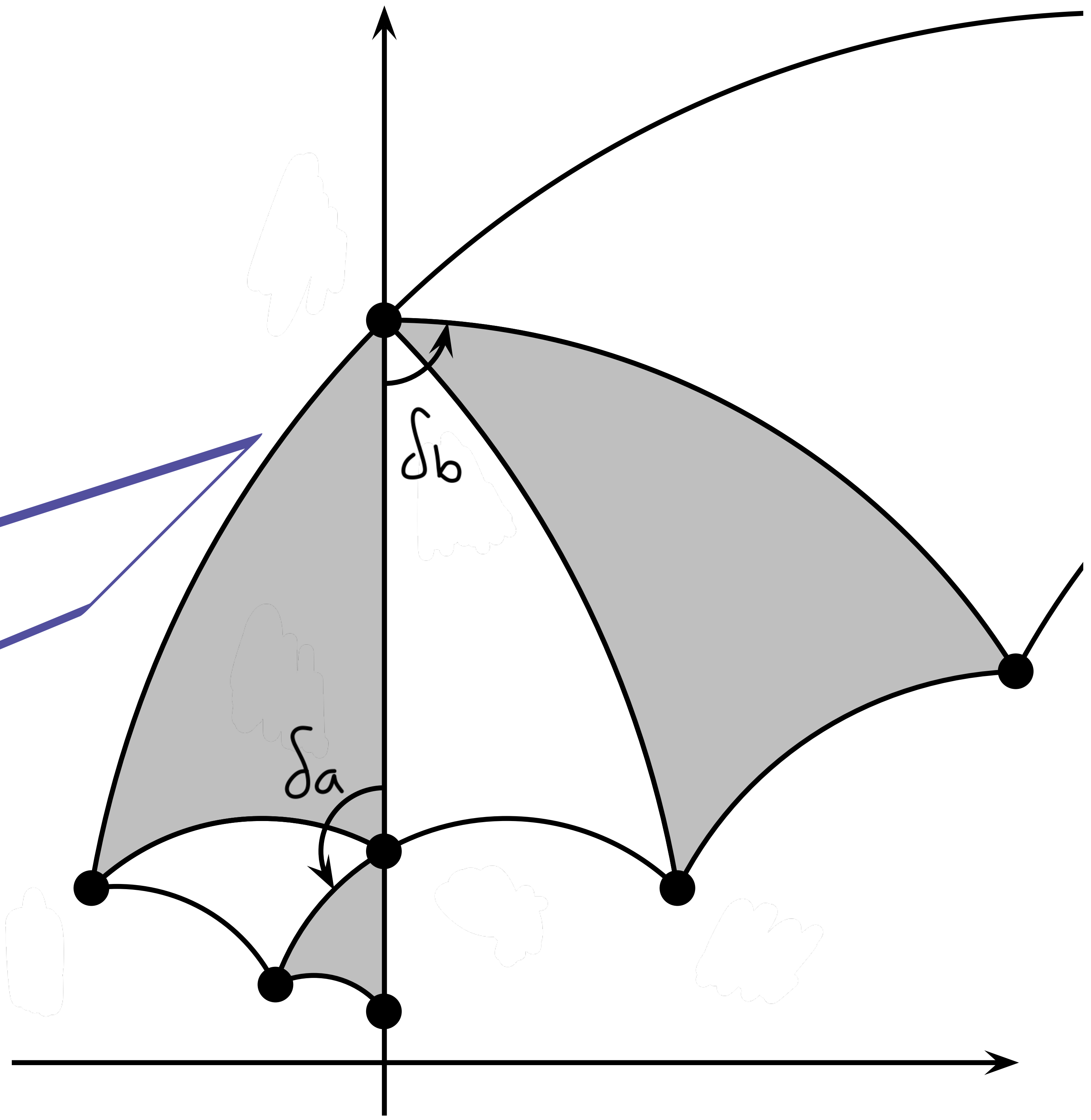
- Given $a, b, c \in \mathbb{Z}_{\geq 2} \cup \{\infty\}$, a triangle group can be presented as:

$$\Delta(a,b,c) := \langle \delta_a, \delta_b, \delta_c \,|\, \delta_a^a = \delta_b^b = \delta_c^c = \delta_a\delta_b\delta_c = 1 \rangle.$$

- There is an action of $\Delta(a,b,c)$ on $\mathscr{H}$.
- A **triangular modular curve** (TMC) is an algebraic curve given by the quotient

$$X(1) = X(a,b,c;1) := \Delta(a,b,c)\backslash\mathscr{H} \simeq \mathbb{P}^1.$$

- Analogously to modular curves, we can define level structure and congruence subgroups [Clark and Voight '19].



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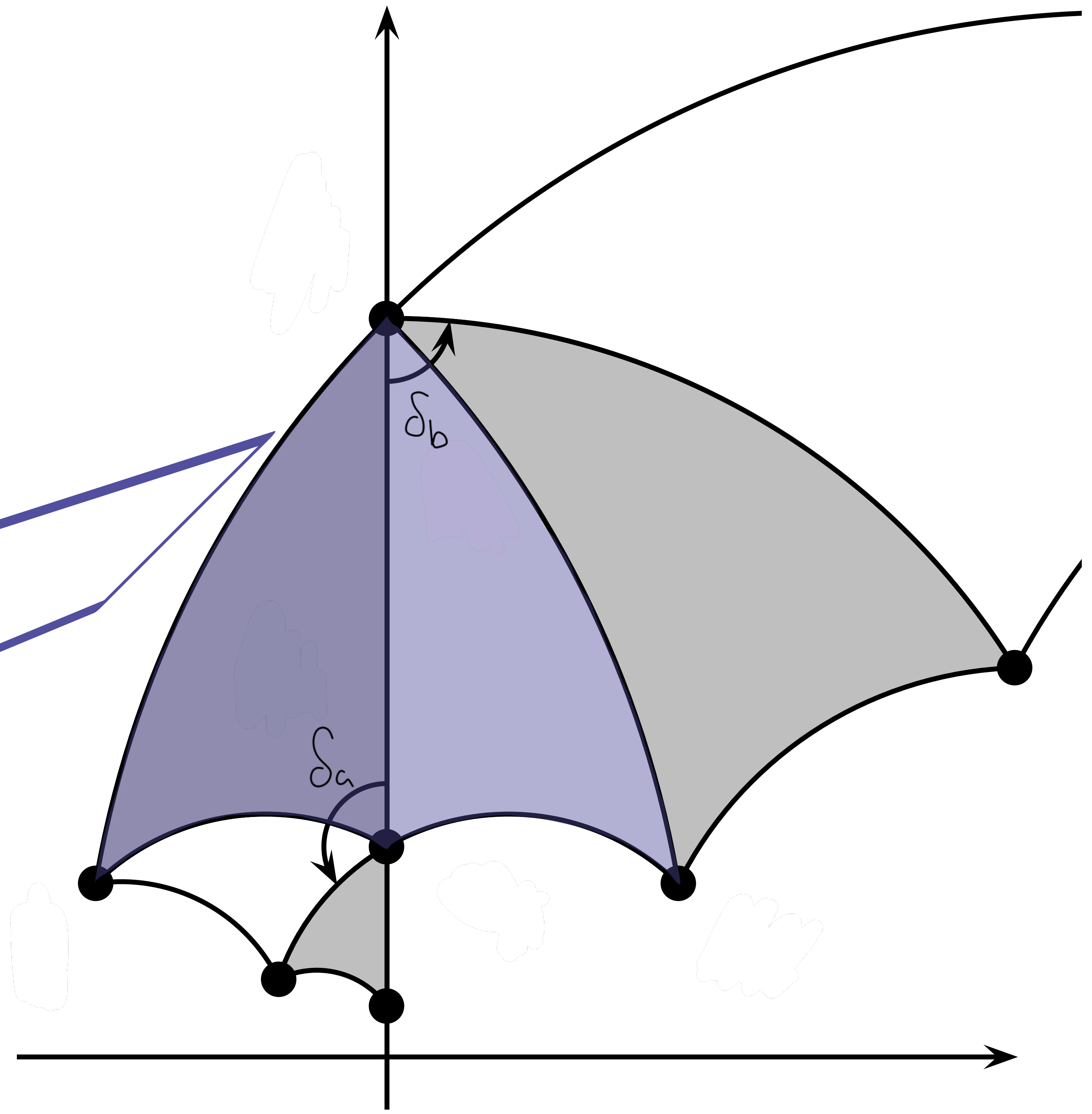

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Motivation

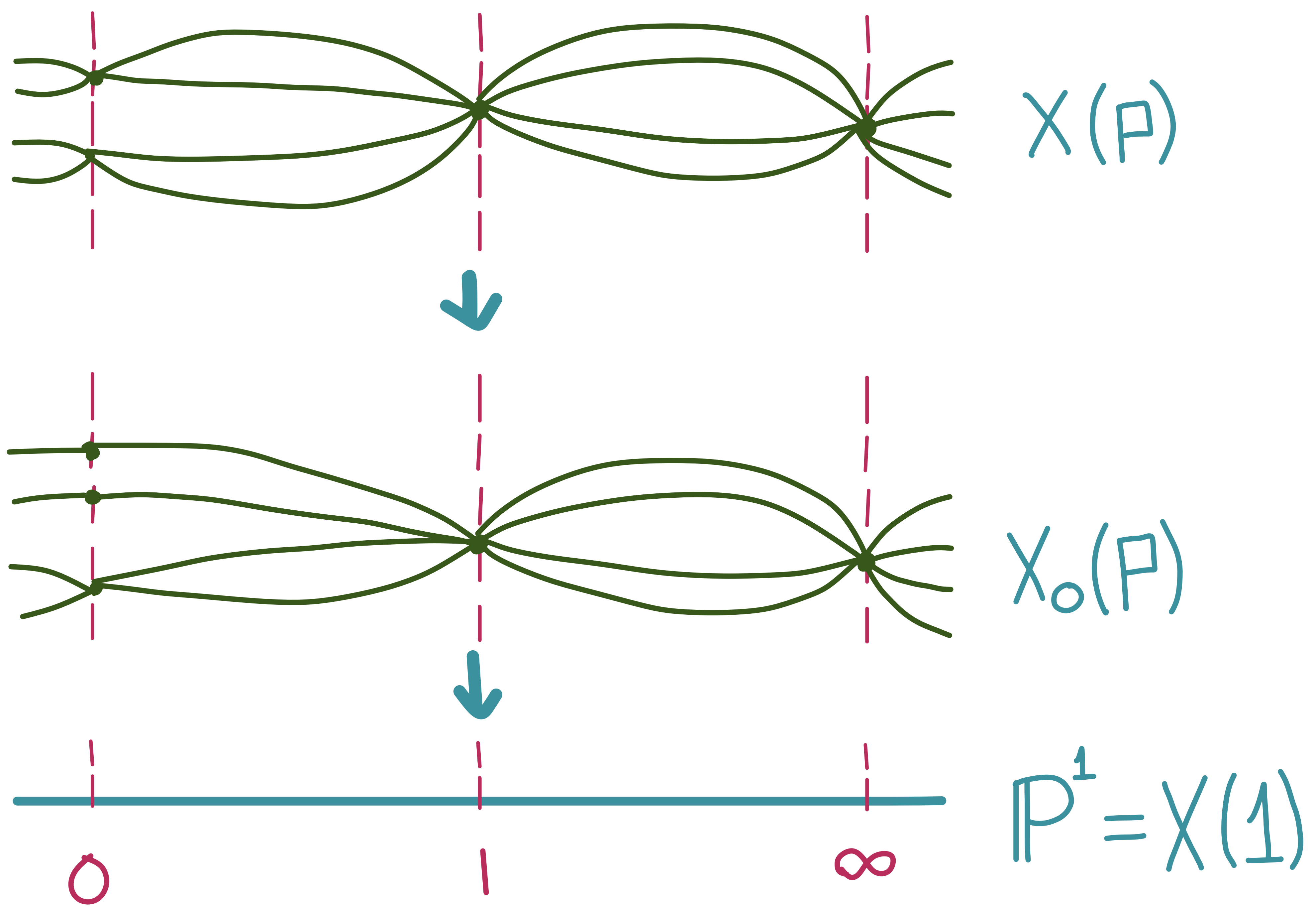
- $\Delta(2,3,\infty;\mathfrak{p}) \cong \mathrm{PSL}_2(\mathbb{F}_p)$.
- **Darmon's program '04:** there is a dictionary between finite index subgroups of the triangle group $\Delta(a,b,c)$ and approaches to solve the generalized Fermat equation

$$x^a + y^b + z^c = 0.$$

- For $t \neq 0, 1, \infty$, consider the family of curves:

$$X_t : y^m = x^{e_0}(x-1)^{e_1}(x-t)^{e_t}.$$

- **Cohen & Wolfart '90, Archinard '03:** the family $\mathrm{Prym}(X_t)$ extends to a family of abelian varieties that are parameterized by triangular modular curves over $\mathbb{P}^1$.

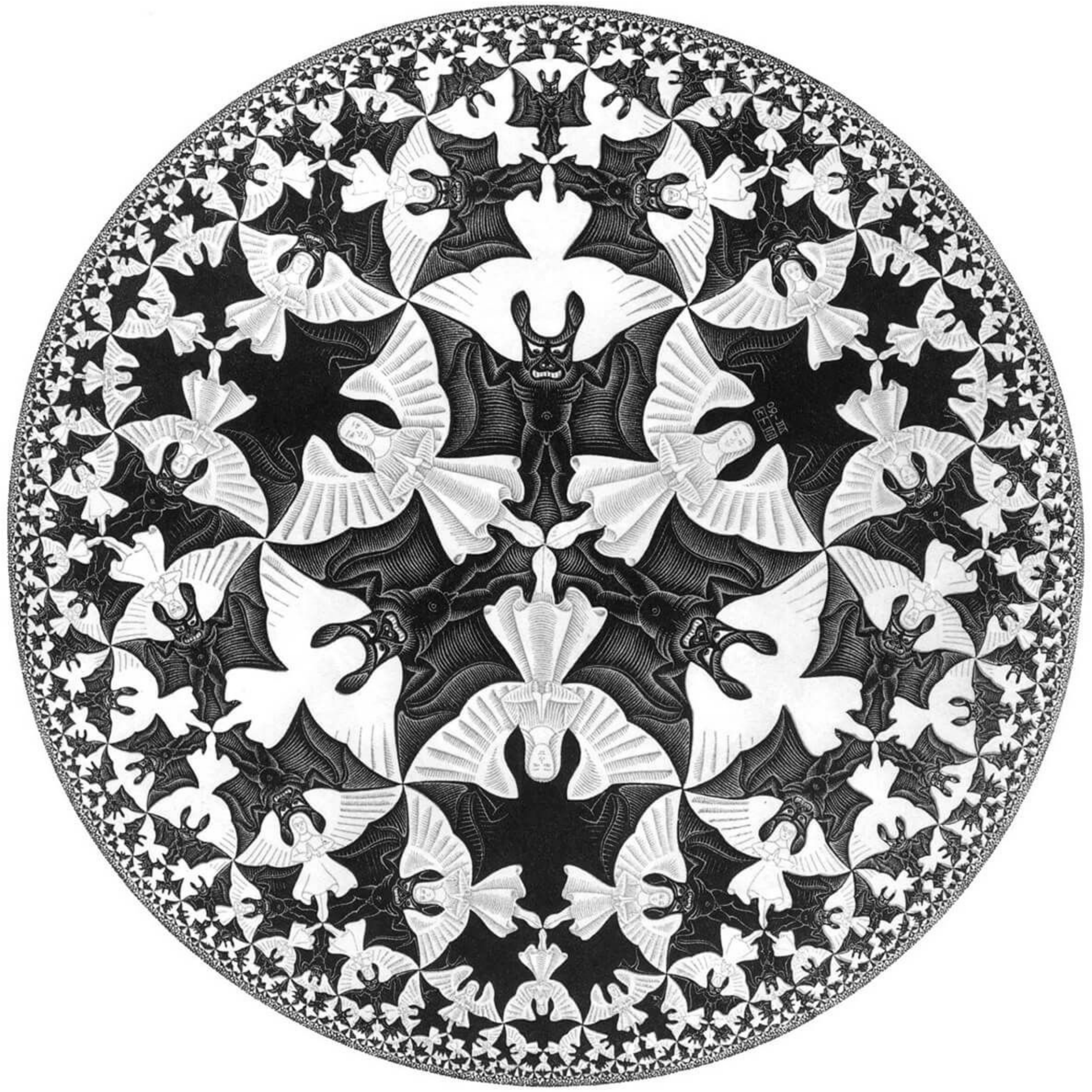


Theorem [DR & Voight]. For any $g \in \mathbb{Z}_{\geq 0}$ there are **finitely many** triangular modular curves $X_0(a, b, c; \mathfrak{N})$ and $X_1(a, b, c; \mathfrak{N})$ of genus g with nontrivial level $\mathfrak{N}$.

Moreover, we present an algorithm to list all such curves of a given genus.

- We first show the result for prime level [DR & Voight '22].
- We use the Riemann-Hurwitz formula on the cover $X_0(a, b, c; \mathfrak{N}) \to \mathbb{P}^1$.
- The monodromy of $X_0(a, b, c; \mathfrak{N}) \to \mathbb{P}^1$ can be computed using optimal embedding numbers.
- We first need to prove the theorem for genus $g = 0, 1$ and the rest of the theorem follows from this.

Future work



Escher: *Angels and Devils*

1. Find models of TMCs of low genus and relate them to the existing database of curves in the LMFDB (at least over $\mathbb{Q}$).
2. Describe all rational points (over the field of definition) of TMCs.
3. **Conjecture.** For all $g \in \mathbb{Z}_{\geq 0}$, there are only finitely many admissible triangular modular curves of genus g.