# Local heights on hyperelliptic curves for quadratic Chabauty.



Joint work with Alex Betts, Sachi Hashimoto, and Pim Spelier.

$C$ will be a nice curve defined over $\mathbb{Q}$.
nice ↓ smooth, projective, geometrically irreducible.

## 1 Rational points

Goal: $C(\mathbb{Q})$.

$g :=$ genus of $C$. $r :=$ Mordell-Weil rank of $\underbrace{J(\mathbb{Q})}_{\text{Finitely generated abelian group.}} = \text{Torsion} \oplus \mathbb{Z}^r$.

Fix basepoint $b \in C(\mathbb{Q})$ and $AJ_b : C \longrightarrow J$.
$Q \longmapsto Q - b$

Earlier results:

- Faltings's Theorem '83: if $g \geq 2$, then $\#C(\mathbb{Q}) < \infty$.
- When $r < g$: Chabauty '41, Coleman '85: If $g \geq 2$, then $\#C(\mathbb{Q}) < \infty$ computable.

Today: Explicit computations for $r = g$.

## 2 Heights and quadratic Chabauty

The p-adic Nekovář height function $h_Z : C(\mathbb{Q}) \longrightarrow \mathbb{Q}_p$ decomposes as

$$h_Z(Q) = \sum_{\ell} h_{Z,\ell}(Q)$$

where $h_{Z,\ell} : C(\mathbb{Q}_\ell) \longrightarrow \mathbb{Q}_p$

Facts:
- For $\ell \neq p$, $h_{Z,\ell}$ takes only finitely many values.
- If $\ell \neq p$ is of potential good reduction, then $h_{Z,\ell} = 0$.
- $h_{Z,\ell}$ can be defined using intersection-pairings on a regular model:

$$h_{Z,\ell}(x) := (x-b) \cdot D_Z(b,x)$$

- $h_{Z,p}$ is locally analytic. Methods for computing it: Balakrishnan-Besser '12, Müller '14, Balakrishnan-Dogra '18, etc.

(A curve $C/\mathbb{Q}$ has <u>bad reduction</u> at $p$ if the reduction of $C$ mod $p$ is singular)

$K$ finite extension of $\mathbb{Q}_\ell$. $C/K$ smooth projective curve

$Z$ correspondence given by a divisor $D$ in $C \times C$.

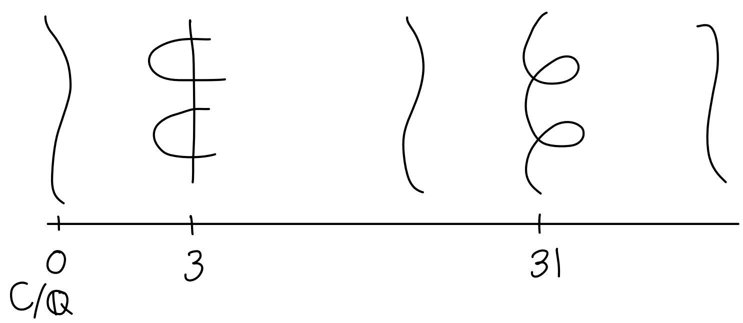
$x \in C(K)$ $\qquad D_Z(b,x) := D|_\Delta - D|_{\{b\}\times C} - D|_{C \times \{x\}}$

$\mathcal{C}$ regular model $\mathcal{D}_Z(b,x) \subset \mathcal{C}$ divisor whose generic fibre is $D_Z(b,x)$.

$$h_{Z,\ell}(x) := (x-b)\cdot \mathcal{D}_Z(b,x) \in \mathbb{Q}.$$

(* Finding explicit equations for a regular model is not great. Working with these equations is not doable in practice.
Even in a genus 2 curve $X_0(67)^+$ $y^2 + (x^3+x+1)y = x^5 - x$, the equations for $D$ have degree 25.
This has not been used for quadratic Chabauty.)

**Example.** Shimura curve $X_0(93,1)/w_{93}$ $\qquad$ Abelian surfaces with potential quaternionic mult.
$C_1 : y^2 = x^6 + 2x^4 + 6x^3 + 5x^2 - 6x + 1$.




regular model for $C_1/\mathbb{Z}$.

**Theorem** [Balakrishnan - Dogra, '18]. (Quadratic Chabauty) $C/\mathbb{Q}$ nice, $g \geq 2$. Assume that $r = g$. Let $p$ be a prime of good reduction. Let $Z \subset C \times C$ be a trace 0 correspondence fixed by the Rosati involution. (Respects the principal polarization).
Then there exists a quadratic function

$$\eta_Z : \mathrm{Lie}(J_{\mathbb{Q}_p}) \longrightarrow \mathbb{Q}_p$$

for which $C(\mathbb{Q})$ is contained in the locus inside $C(\mathbb{Q}_p)$ cut out by the equations $\eta_Z(\log(x)) - h_{Z,p}(x) \in \Omega$, 3

where $\Omega$ is the finite set:

$$\Omega = \Big\{ \sum_{\ell \neq p} h_{Z,\ell}(x_\ell) : x_\ell \in C(\mathbb{Q}_\ell) \Big\}.$$

Application. [BDMTV, '19] Rational points on the "cursed curve" $X_{ns}(13)$.

non-split Cartan mod. curve. genus 3.

Note: $X_{ns}(13) \simeq X_s(13)$.

$X_s(13)$ has potential good reduction everywhere, so $h_{Z,\ell} = 0$ $\forall \ell \neq p$.

(they choose $p=17$).

Applications to Serre's uniformity conjecture for Galois representations of elliptic curves.

## 3 Computing local heights

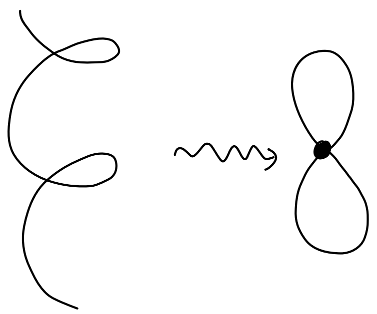
Theorem [Betts - DR - Hashimoto - Spelier, '24] Let $C/\mathbb{Q}$ be a hyperelliptic curve $y^2 = f(x)$ of genus $g \geq 2$. Then there is an explicit/practical combinatorial method for computing $h_{Z,\ell}$ where $\ell \neq p, 2$.

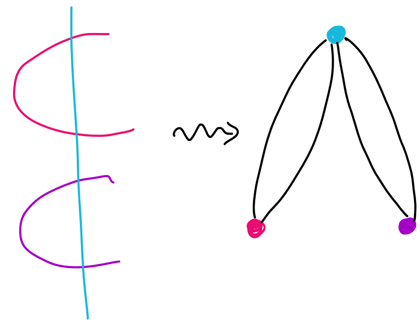
Def. A split semistable model is geometrically reduced + at worst double points as singularities + every component is geometrically irred, every singular pt is $k$-rat + tangent direc. @ singular pts are $k$-rational.

**Reduction graph $\Gamma$:** Given a split semistable model $\mathcal{C}$

- Vertices: irreducible components of the special fiber.
- Edges: singular points of $\mathcal{C}$.

**Examples:**





**Theorem [Betts-Dogra, '20]** $K$ finite ext. of $\mathbb{Q}_\ell$, $C/K$ nice
$Z \subset C \times C$ correspondence of trace 0 fixed by the Rosati involution.
$\Gamma$ reduction graph of a split semistable model of $C$. Then there is a formula for $h_{Z,\ell}: C(K) \to \mathbb{Q}$ that uses:

- $Z_*: H_1(\Gamma, \mathbb{Z}) \to H_1(\Gamma, \mathbb{Z})$.
- $tr_v(Z)$ $\forall v \in V(\Gamma)$. $= 0$ for genus $0$ components.

$h_{Z,\ell}: C(K) \to \mathbb{Q}$ factors through a piecewise polynomial function $h_{Z,\ell}: \Gamma \to \mathbb{R}$

(with Laplacian

$$\nabla^2(h_{Z,\ell}) = 2 \sum_{e \in E(\Gamma)^+} \frac{1}{\ell(e)} \langle e, Z_*(\pi(e)) \rangle \cdot |ds_e| + \sum_{v \in V(\Gamma)} tr_v(Z) \cdot \delta_v$$

$$\nabla^2(f) := -\sum_e (f|_e)'' \cdot |ds_e| - \sum_v \left( \sum_{\vec{v} \in T_v(\Gamma)} D_{\vec{v}} f(v) \right) \cdot \delta_v.$$
)

## 3 Our method.

### STEP 1. Cluster pictures and Semistable coverings

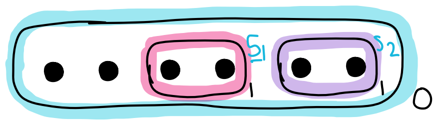
Split semistable model for $C$ $\xrightarrow{\text{Coleman}}$ semistable covering of $C^{an}$ — the rigid-analytification of $C$

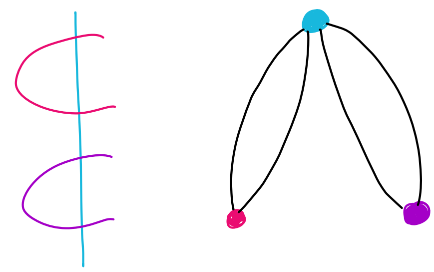
simplest conceptually | Computations

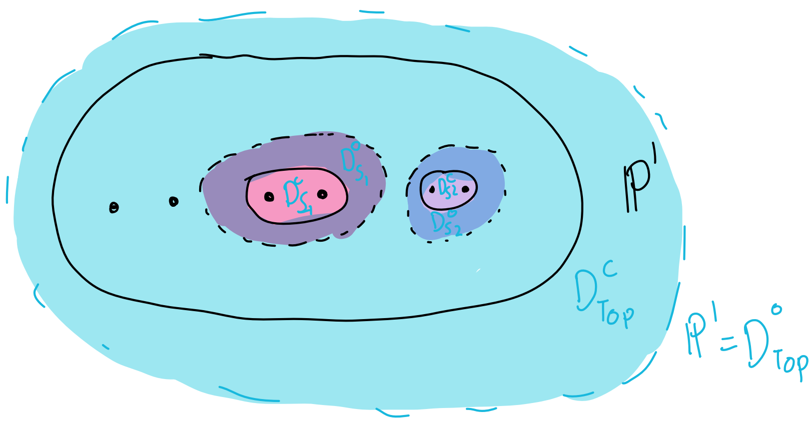
**Theorem [Betts-DR-Hashimoto-Spelier, '24]** $C/K$ hyperelliptic curve with split semistable reduction given by $y^2=f(x)$. Then the cluster picture of $C$ gives an explicit semistable covering of $C^{an}$.

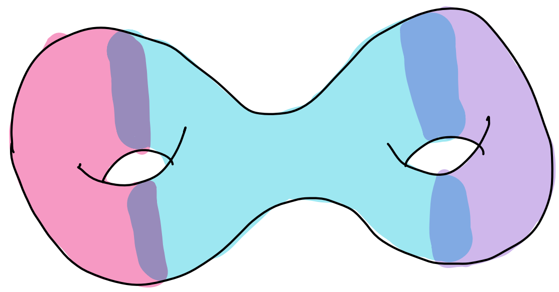
**Example.** $C_1$ as before. $\ell=3$ $\quad C_1: y^2 = \deg 6$ polynomial.

modulo 9: $\quad y^2=(x-2)^2(x-1)^2(x^2-3x-2)$





**STEP 2.** Coleman–Iovita isomorphism

Need to compute $Z_*$ on $H_1(\Gamma,\mathbb{Z})$.

**Theorem [Coleman–Iovita '10 + Darmon–Rotger 17]**

The map

$$H^0(C_{\mathbb{Q}_\ell},\Omega^1_C) \longrightarrow H_1(\Gamma,\mathbb{Q}_\ell)$$
$$\omega \longmapsto \sum_{e\in E(\Gamma)} \mathrm{Res}_{A_{\vec e}}(\omega)\cdot\vec e$$

is surjective. It is an isomorphism if every component of the $\ell$-adic special fiber has genus 0.

**Lemma.** The isomorphism behaves well wrt $Z_*$.

We know $Z_*$ on $H_1(\Gamma,\mathbb{Q}_\ell)$ up to any $\ell$-adic precision.

## STEP 3. Bounds

**Theorem** [Betts-DR-Hashimoto-Spelier '24].
Let $Z$ be an effective correspondence of degrees $d_1$ and $d_2$.
Then $Z_* : H_1(\Gamma, \mathbb{Z}) \longrightarrow H_1(\Gamma, \mathbb{Z})$ has operator norm $\leq \sqrt{d_1 d_2}$.
(wrt intersection length pairing).

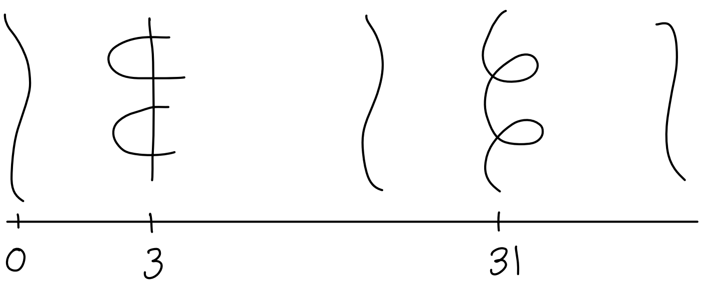
Then computing up to finite precision is enough!

## ④ Examples.

(a) $C_1 : y^2 = x^6 + 2x^4 + 6x^3 + 5x^2 - 6x + 1$.

Shimura curve $X_0(93,1)/w_{93}$

Exceptionally isomorphic to $X_0(93)^*$
RM multiplication by $\sqrt{5}$.

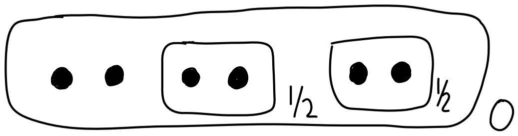



STEP 2: Obtain $\begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix} + O(3^4)$.

STEP 3: Correspondence has degrees 2 and 10, so we get a bound of $16\sqrt{5} + 2 < 38 < 81$.

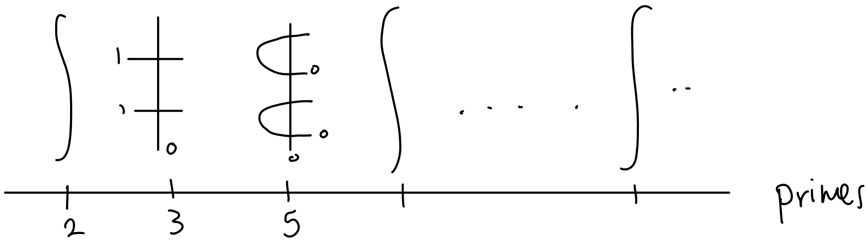
Apply to the B-D formula:

$$h_{Z,3}([x:y:z]) = \begin{cases} \frac{1}{2} & \frac{x}{z} \equiv 1 \mod 3 \\ -\frac{1}{2} & \frac{x}{z} \equiv 2 \mod 3 \\ 0 & \text{otherwise.} \end{cases}$$

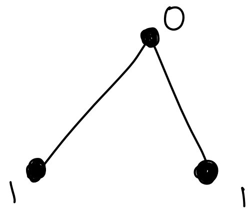
(b) 

has heigths equal to zero.

(c) $C_2 : y^2 = x^6 + \frac{18}{5}x^4 + \frac{6}{5}x^3 + \frac{9}{5}x^2 + \frac{6}{5}x + \frac{1}{5}$. RM by $\sqrt{13}$.

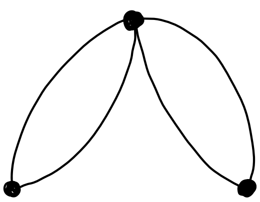



$l = 3$




$$h_{Z,3}([x:y:z]) = \begin{cases} -\frac{2}{3} & \frac{x}{z} \equiv 1 \mod 3 \\ 0 & \text{otherwise} \end{cases}$$

$l=5$



$$h_{Z,5}([x:y:z]) = \begin{cases} \frac{3}{4} & z \equiv 0 \mod 5 \\ -\frac{3}{4} & \frac{x}{z} \equiv 2 \mod 5 \\ 0 & \text{otherwise} \end{cases}$$

**Theorem [BDRHS, '24]** The curve $C_2$ has exactly 10 rational points.