ENUMERATING TRIANGULAR MODULAR CURVES OF LOW GENUS

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Theorem (DR & Voight, 2022)

For any $g \in \mathbb{Z}_{\geq 0}$, there are only finitely many Borel-type triangular modular curves $X_0(a, b, c; \mathfrak{N})$ and $X_1(a, b, c; \mathfrak{N})$ of genus g with nontrivial level $\mathfrak{N} \neq (1)$. The number of curves $X_0(a, b, c; \mathfrak{N})$ of genus ≤ 2 are as follows:

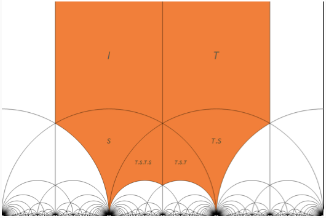
- *71 curves of genus 0,*
- *190 curves of genus 1.*
- *153 curves of genus 2.*

We consider the Legendre family of elliptic curves

$$y^2 = x(x-1)(x-\lambda)$$

for a parameter $\lambda \neq 0, 1, \infty$.

- A curve in this family is a cyclic cover of $\mathbb{P}^1$ branched at 4 points.
- We can parameterize the family by the modular curve $X(2) = \mathbb{P}^1$.
- One can study additional level structure by considering covers of $X(2)$, specifying extra data such as a cyclic N-isogeny or an N-torsion point.



Fundamental domain for $\Gamma(2)$, by Paul Kainberger.

We consider the family of curves

$$X_t : y^m = x^{e_0}(x-1)^{e_1}(t-x)^{e_t},$$

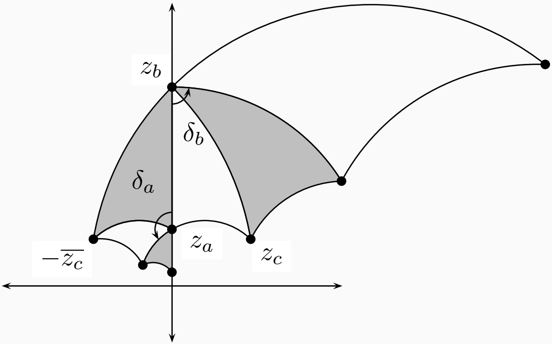
where $t \neq 0, 1, \infty$.

- A curve in this family is a cyclic cover of $\mathbb{P}^1$ branched at 4 points.
- X_t has a cyclic group of automorphisms of order m, defined over $\mathbb{Q}(\zeta_m)$, given by $\alpha : (x, y) \mapsto (x, \zeta_m y)$.
- We study $\mathrm{Prym}(X_t)$, an isogeny factor of $\mathrm{Jac}(X_t)$ where α acts by a primitive m-th root of unity.

By work of Cohen & Wolfart (1990), triangular modular curves parameterize the varieties $\mathrm{Prym}(X_t)$.

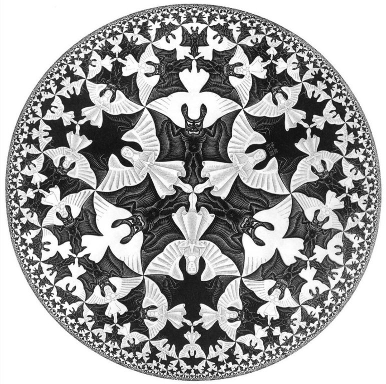
For $a, b, c \in \mathbb{Z}_{\geq 2} \cup \{\infty\}$, let $\Delta(a, b, c)$ be the **triangle group** with presentation

$$\Delta = \Delta(a, b, c) := \langle \delta_a, \delta_b, \delta_c \mid \delta_a^a = \delta_b^b = \delta_c^c = \delta_a\delta_b\delta_c = 1 \rangle.$$



The triple $(a, b, c) \in (\mathbb{Z}_{\geq 2} \cup \{\infty\})^3$ is **hyperbolic** if and only if

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} - 1 < 0.$$



Escher, Angels and Demons.

We now fix a hyperbolic triple $(a, b, c) \in \mathbb{Z}_{\geq 2} \cup \{\infty\}$.

Theorem (Takeuchi, 1977)

There is an embedding

$$\Delta \hookrightarrow \mathrm{SL}_2(\mathbb{R})$$

that can be given explicitly in terms of the functions $\sin(\pi/s)$ *and* $\cos(\pi/s)$. *This embedding is unique up to conjugacy in* $\mathrm{SL}_2(\mathbb{R})$.

The quotient

$$X(1) := \Delta \backslash \mathcal{H}$$

is a complex Riemannian 1-orbifold of genus zero. We call this quotient a **triangular modular curve (TMC)**.

Let p be a rational prime with $p \nmid 2abc$. We consider the tower of fields

$$F := \mathbb{Q}\left(\cos\left(\tfrac{\pi}{a}\right), \cos\left(\tfrac{\pi}{b}\right), \cos\left(\tfrac{\pi}{c}\right)\right) \qquad \mathfrak{P}$$

$$E := \mathbb{Q}\left(\cos\left(\tfrac{2\pi}{a}\right), \cos\left(\tfrac{2\pi}{b}\right), \cos\left(\tfrac{2\pi}{c}\right), \cos\left(\tfrac{\pi}{a}\right)\cos\left(\tfrac{\pi}{b}\right)\cos\left(\tfrac{\pi}{c}\right)\right) \qquad \mathfrak{p}$$

$$\mathbb{Q} \qquad p$$

Remark: $2\cos\left(\frac{2\pi}{s}\right) = \zeta_s + 1/\zeta_s$, where $\zeta_s := \exp(2\pi i/s)$.

There is a homomorphism

$$\phi_{\mathfrak{P}} : \Delta \to \mathrm{PSL}_2(\mathbb{Z}_F/\mathfrak{P})$$

We let the subgroup $\Delta(\mathfrak{p})$ be the subgroup given by $\ker \phi_{\mathfrak{P}} \subseteq \Delta$.

Theorem (Clark & Voight, 2019)

Let (a, b, c) *be a hyperbolic triple with* $a, b, c \in \mathbb{Z}_{\geq 2} \cup \{\infty\}$. *Let* $\mathfrak{p}$ *be a prime of* E *with residue field* $\mathbb{F}_{\mathfrak{p}}$ *and suppose* $\mathfrak{p} \nmid 2abc$. *Then there exists a* G*-Galois Belyi map* $X(\mathfrak{p}) \to \mathbb{P}^1$ *with ramification indices* (a, b, c), *where*

$$G = \begin{cases} \mathrm{PSL}_2(\mathbb{F}_{\mathfrak{p}}), & \text{if } \mathfrak{p} \text{ splits completely in } F; \\ \mathrm{PGL}_2(\mathbb{F}_{\mathfrak{p}}), & \text{otherwise.} \end{cases}$$

Remark: The genus of $X(\mathfrak{p})$ is given by

$$g(X(\mathfrak{p})) = 1 + \frac{\#G}{2}\left(1 - \frac{1}{a} - \frac{1}{b} - \frac{1}{c}\right).$$

Analogously, we obtain curves $X(\mathfrak{N}) := \Delta(\mathfrak{N})\backslash\mathcal{H}$ for every ideal $\mathfrak{N}$ of E.

Recall the homomorphism

$$\phi_{\mathfrak{N}} : \Delta \to \mathrm{PSL}_2(\mathbb{Z}_F/\mathfrak{N}\mathbb{Z}_F),$$

and that $\Delta(\mathfrak{N}) := \ker \phi_{\mathfrak{N}}$.

Define the subgroups,

$$H_0 := \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}, \qquad H_1 := \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\} \subseteq \mathrm{PSL}_2(\mathbb{Z}_F/\mathfrak{N}\mathbb{Z}_F),$$

$$\Gamma_0(\mathfrak{N}) := \phi_{\mathfrak{N}}^{-1}(H_0), \qquad \Gamma_1(\mathfrak{N}) := \phi_{\mathfrak{N}}^{-1}(H_1) \subseteq \Delta.$$

We define the curves

$$X_0(\mathfrak{N}) := \Gamma_0(\mathfrak{N})\backslash\mathcal{H}, \qquad X_1(\mathfrak{N}) := \Gamma_1(\mathfrak{N})\backslash\mathcal{H}.$$

Then we have maps

$$X(\mathfrak{N}) \to X_1(\mathfrak{N}) \to X_0(\mathfrak{N}) \to X(1).$$

Example

We consider the triple $(a,b,c) = (2,3,\infty)$. In this case we have that $\mathbb{Q} = E = F$ and $\mathrm{SL}_2(\mathbb{Z}) \cong \Delta(2,3,\infty)$. By construction, $X(2,3,\infty;N) = X(N)$.

Example

Let $p \geq 5$ be a prime. We consider hyperbolic triples of the form $(2,3,c)$ with $c = p^k$ and $k \geq 1$. Then $X(2,3,p^k;p) \cong X(2,3,p;p)$.

Definition

An ideal $\mathfrak{N} \subseteq \mathbb{Z}_E$ is *admissible* for (a,b,c) if $\mathfrak{N}$ is nonzero and the following two conditions hold:

1. $\mathfrak{N}$ is coprime to $\beta(a,b,c)$, and
2. if $\mathfrak{p} \mid \mathfrak{N}$ is a prime lying above $p \in \mathbb{Z}$, and $p \mid s$ for $s \in \{a,b,c\}$, then $p = s$.

Lemma

Let p be an odd prime and $\mathfrak{p}$ be a prime of E above p. Let $\mathfrak{p}$ be admissible for (a, b, c) and let $q = p^r$ be such that $\mathbb{F}_{\mathfrak{p}} = \mathbb{F}_q$. Then the genus of the curve $X_0(\mathfrak{p})$ is given by

$$2g - 2 = -2(q+1) + k_a(a-1) + k_b(b-1) + k_c(c-1),$$

where

$$k_s = \begin{cases} \frac{q-1}{s} & s|(q-1), \\ \frac{q}{s} & s|q, \\ \frac{q+1}{s} & s|(q+1). \end{cases}$$

Corollary

Let p be an odd prime and $\mathfrak{p}$ be a prime of E above p. Let $\mathfrak{p}$ be admissible for (a, b, c) and let $q = p^r$ be such that $\mathbb{F}_{\mathfrak{p}} = \mathbb{F}_q$. Then,

$$g(X_0(\mathfrak{p})) \geq \frac{q-1}{2}\left(1 - \frac{1}{a} - \frac{1}{b} - \frac{1}{c}\right) - 1.$$

If (a, b, c) is a hyperbolic triple, we have

$$1 - \frac{1}{a} - \frac{1}{b} - \frac{1}{c} \geq \frac{1}{42}.$$

Then, for a fixed g_0,

$$q \leq 82g_0 + 1.$$

Theorem (DR & Voight, 2022)

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Idea of the algorithm

This algorithm returns a list *lowGenus* of all hyperbolic triples (a, b, c), primes p and Galois groups $\mathrm{PSL}_2(\mathbb{F}_q)$ or $\mathrm{PGL}(\mathbb{F}_q)$ such that the genus of $X_0(a, b, c; p)$ is less than g_0.

1. Loop over the list of possible powers $q = p^r$, where p is any rational prime and $q \leq 84(g_0 + 1) + 1$.
2. For each q, find all hyperbolic triples (a, b, c) for which $\mathfrak{p}$ is (a, b, c)-admissible with $\mathbb{F}_\mathfrak{p} = \mathbb{F}_q$.
3. For each such triple (a, b, c), compute the genus g of $X_0(a, b, c; \mathfrak{p})$. If $g \leq g_0$, add $(a, b, c; p, q)$ to the list *lowGenus*.

Future work

- Address the existence of triangular modular curves (and their genera) without the admissibility hypothesis.
- Find models and compute rational points of TMCs of low genus.
- Study TMCs for all congruence subgroups of triangle groups.

Code is available at GitHub.

Curves $X_0(a, b, c; \mathfrak{p})$ of genus 0

(a,b,c)	p
$(2,3,7)$	7
$(2,3,7)$	2
$(2,3,7)$	13
$(2,3,7)$	29
$(2,3,7)$	43
$(2,3,8)$	7
$(2,3,8)$	3
$(2,3,8)$	17
$(2,3,8)$	5
$(2,3,9)$	19
$(2,3,9)$	37
$(2,3,10)$	11
$(2,3,10)$	31
$(2,3,12)$	13
$(2,3,12)$	5
$(2,3,13)$	13
$(2,3,15)$	2
$(2,3,18)$	19

(a,b,c)	p
$(2,4,5)$	5
$(2,4,5)$	3
$(2,4,5)$	11
$(2,4,5)$	41
$(2,4,6)$	5
$(2,4,6)$	7
$(2,4,6)$	13
$(2,4,8)$	3
$(2,4,8)$	17
$(2,4,12)$	13
$(2,5,5)$	5
$(2,5,5)$	11
$(2,5,10)$	11
$(2,6,6)$	7
$(2,6,6)$	13
$(2,6,7)$	7
$(2,8,8)$	3

(a,b,c)	p
$(3,3,4)$	7
$(3,3,4)$	3
$(3,3,4)$	5
$(3,3,5)$	2
$(3,3,6)$	13
$(3,3,7)$	7
$(3,4,4)$	5
$(3,4,4)$	13
$(3,6,6)$	7
$(4,4,4)$	3
$(4,4,5)$	5
$(2,3,\infty)$	2
$(2,3,\infty)$	3
$(2,3,\infty)$	5
$(2,3,\infty)$	7
$(2,3,\infty)$	13
$(2,4,\infty)$	3

(a,b,c)	p
$(2,4,\infty)$	5
$(2,5,\infty)$	5
$(2,5,\infty)$	3
$(2,6,\infty)$	7
$(2,8,\infty)$	3
$(2,\infty,\infty)$	3
$(2,\infty,\infty)$	5
$(3,3,\infty)$	3
$(3,3,\infty)$	7
$(3,4,\infty)$	3
$(3,7,\infty)$	2
$(3,15,\infty)$	2
$(3,\infty,\infty)$	2
$(3,\infty,\infty)$	3
$(4,4,\infty)$	5
$(4,\infty,\infty)$	3
(∞,∞,∞)	3