Triangular modular curves of low genus

Juanita Duque-Rosero

Joint work with John Voight

AMS Special Session on Latinx and Hispanics in Combinatorics, Number Theory, Geometry and Topology

October 2022

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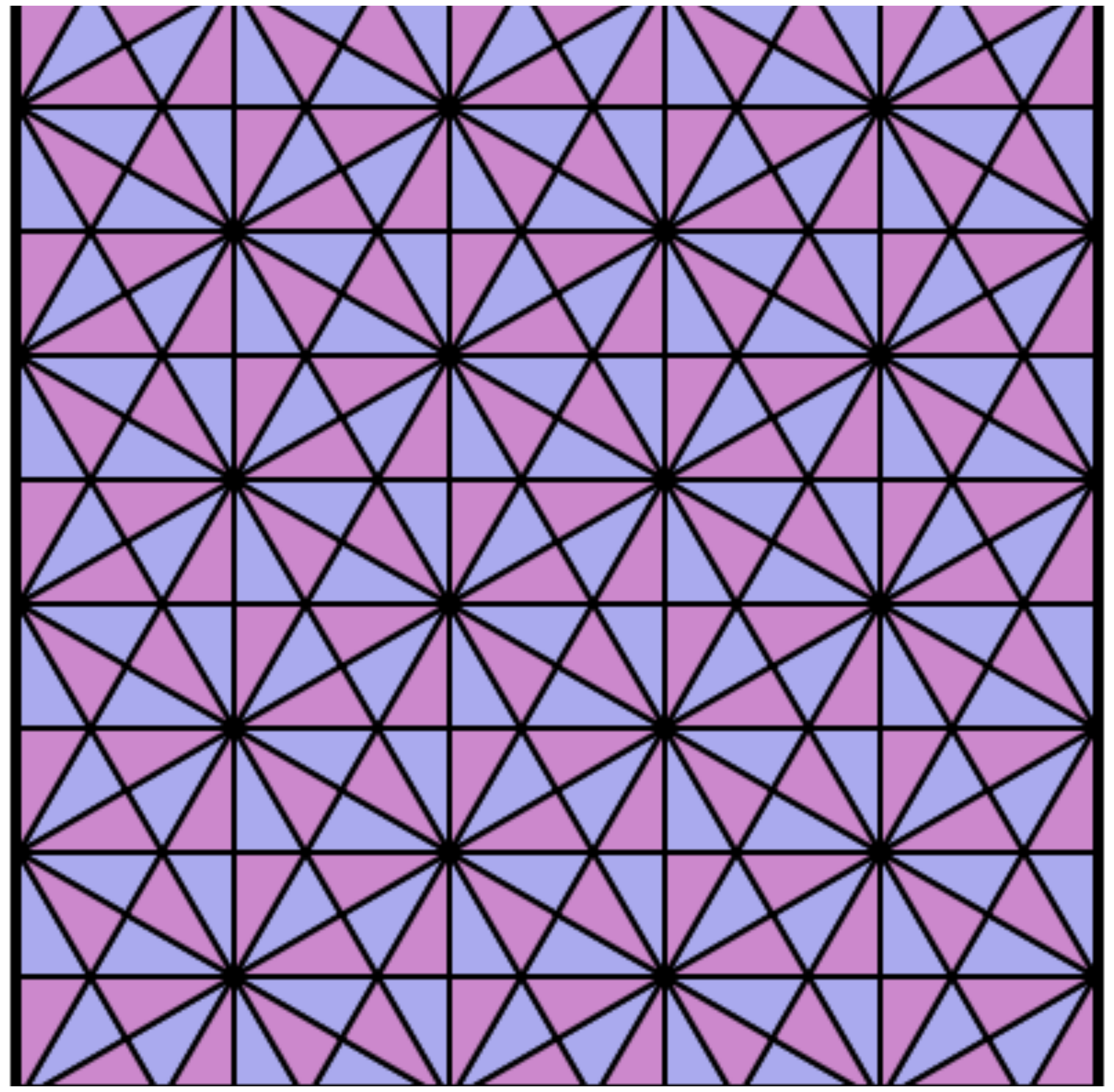
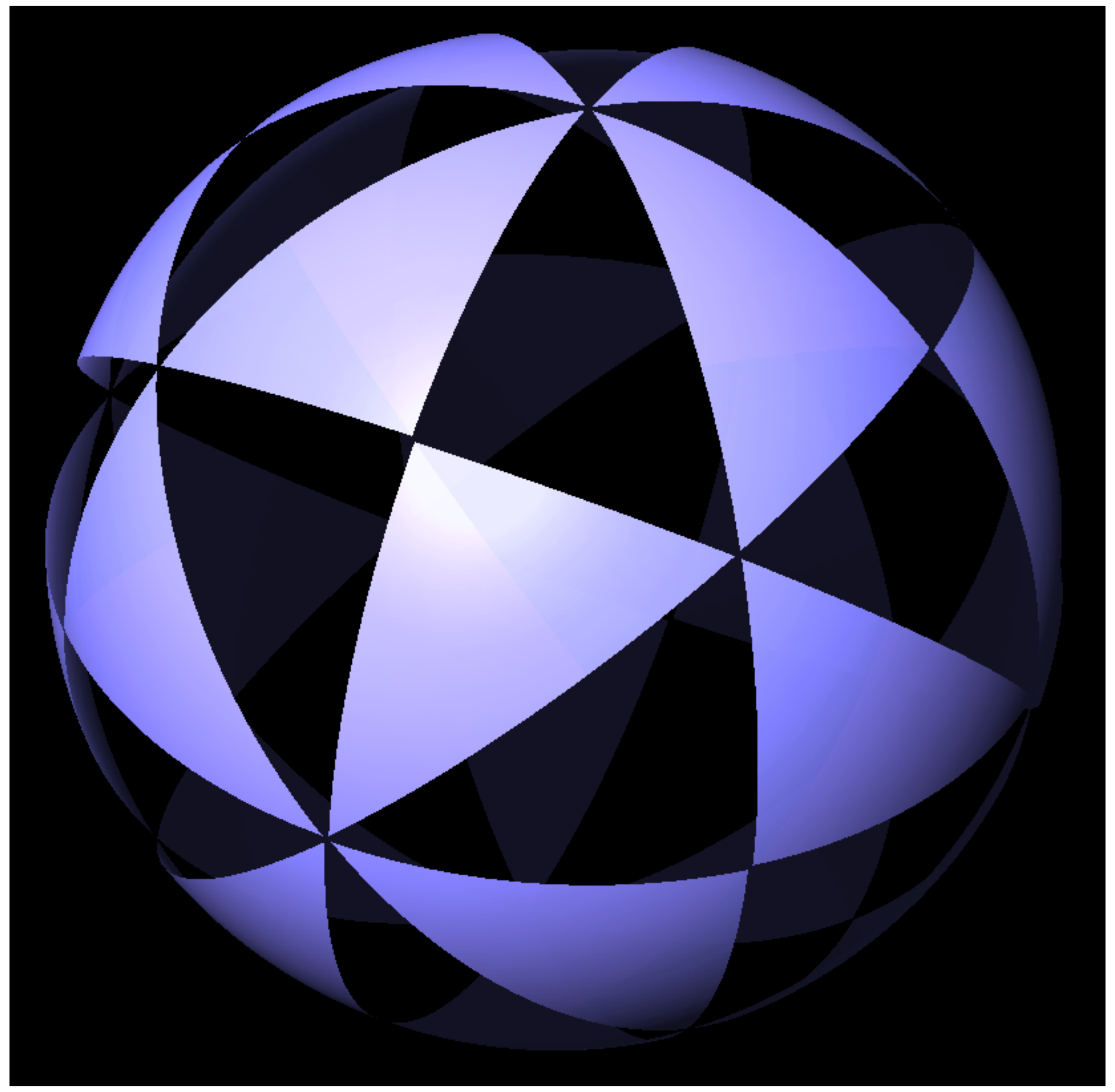
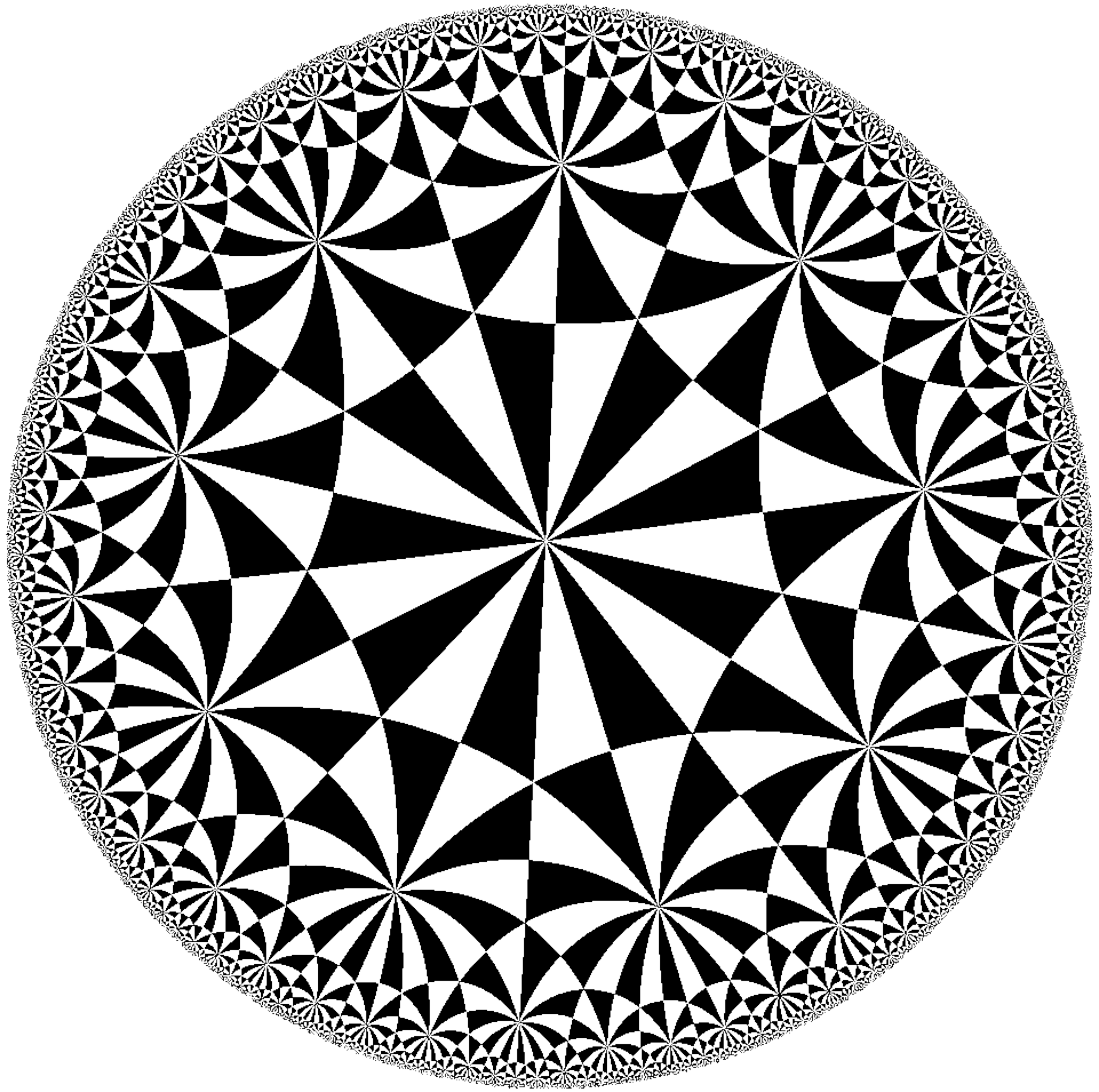
I am on the job market!

Triangle groups
Examples


(2,3,6)


(2,3,4)


(2,3,7)

Triangle groups

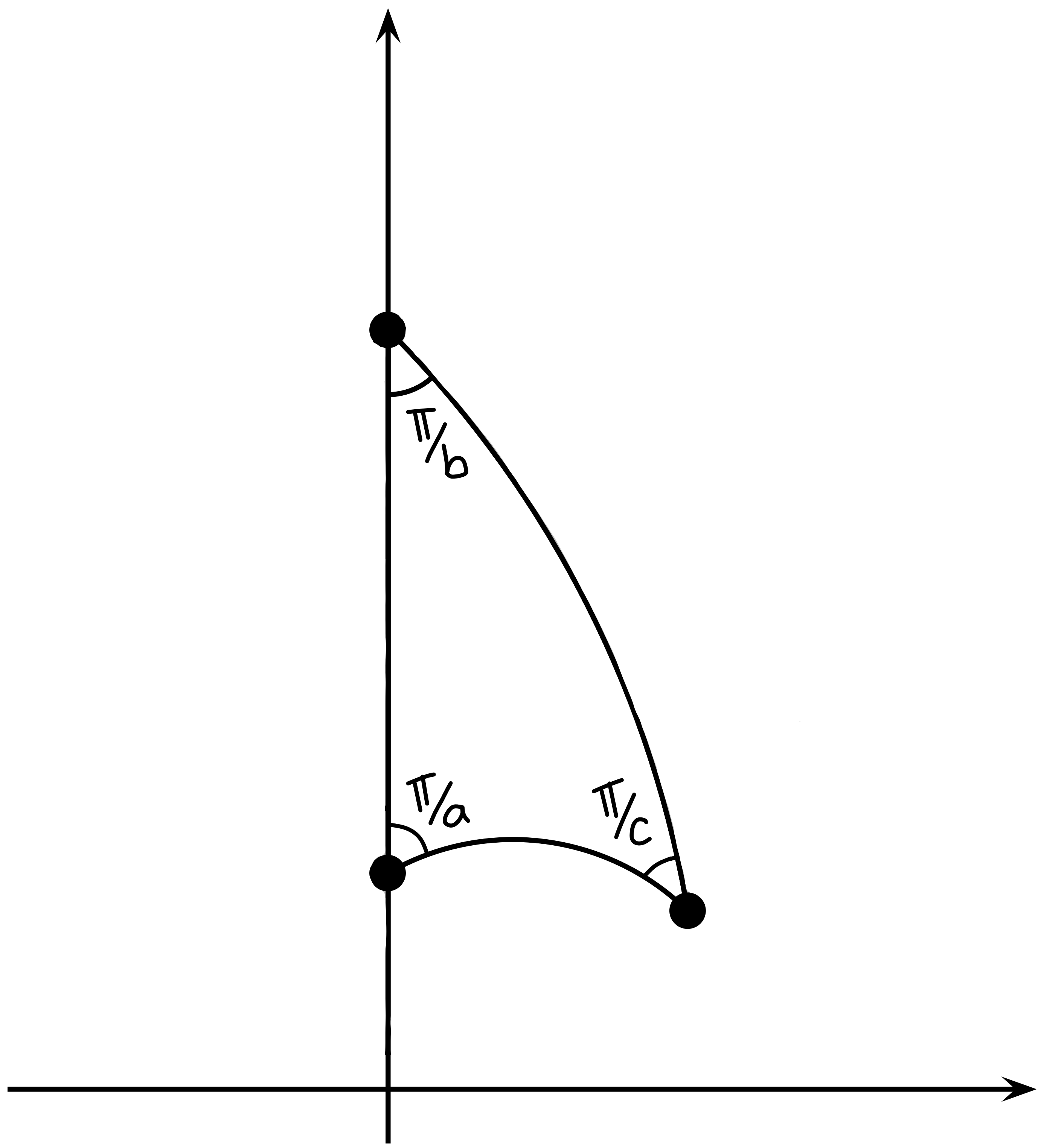
Definition

Let $a, b, c \in \mathbb{Z}_{\geq 2} \cup \{\infty\}$. The **triangle group** is a group with presentation:

$$\Delta(a,b,c) := \langle \delta_a, \delta_b, \delta_c \mid \delta_a^a = \delta_b^b = \delta_c^c = \delta_a\delta_b\delta_c = 1 \rangle$$

We only consider hyperbolic triangles. This is the triple (a,b,c) is hyperbolic:

$$\chi(a,b,c) := \frac{1}{a} + \frac{1}{b} + \frac{1}{c} - 1 < 0$$



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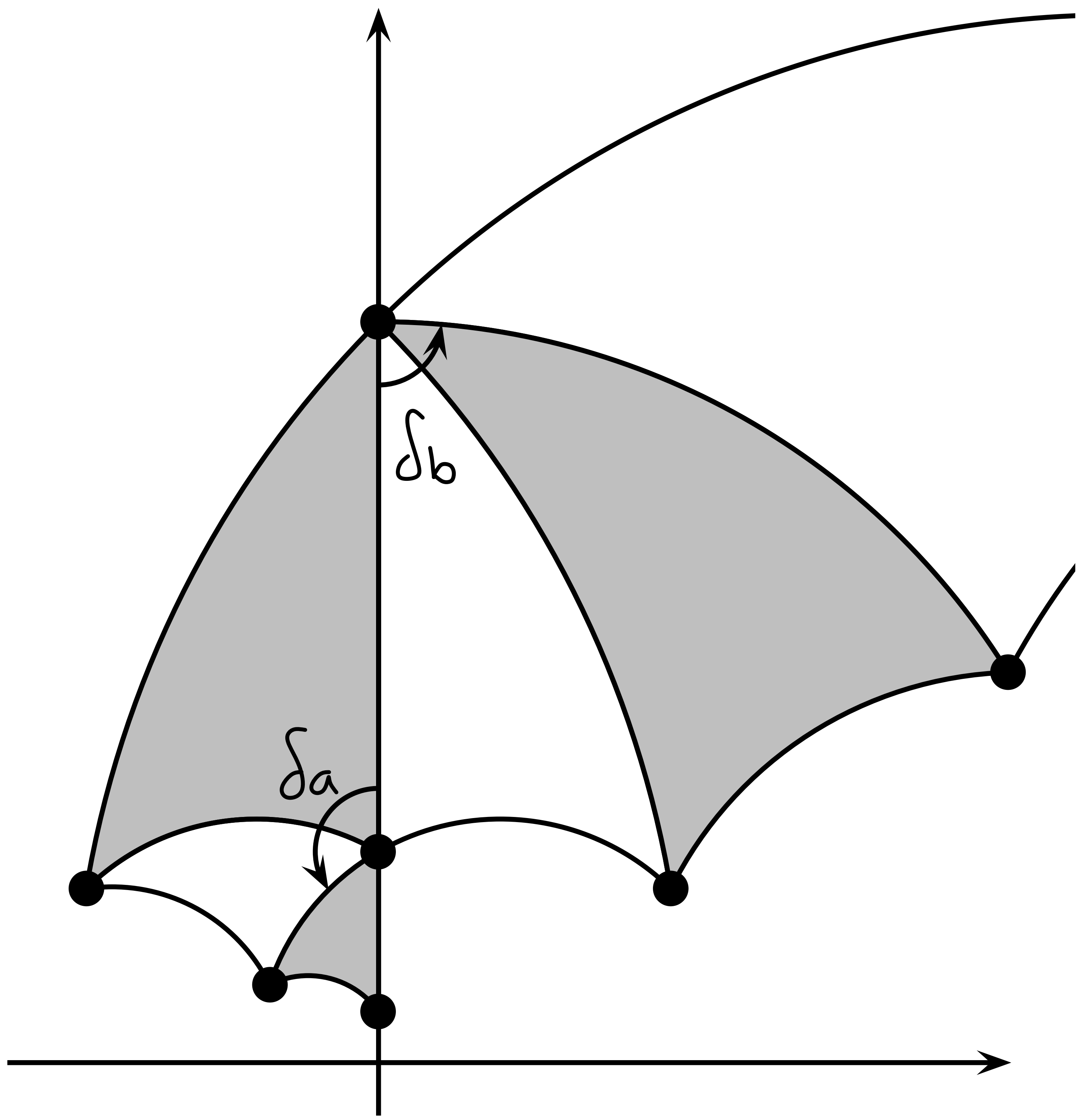


Triangular modular curves

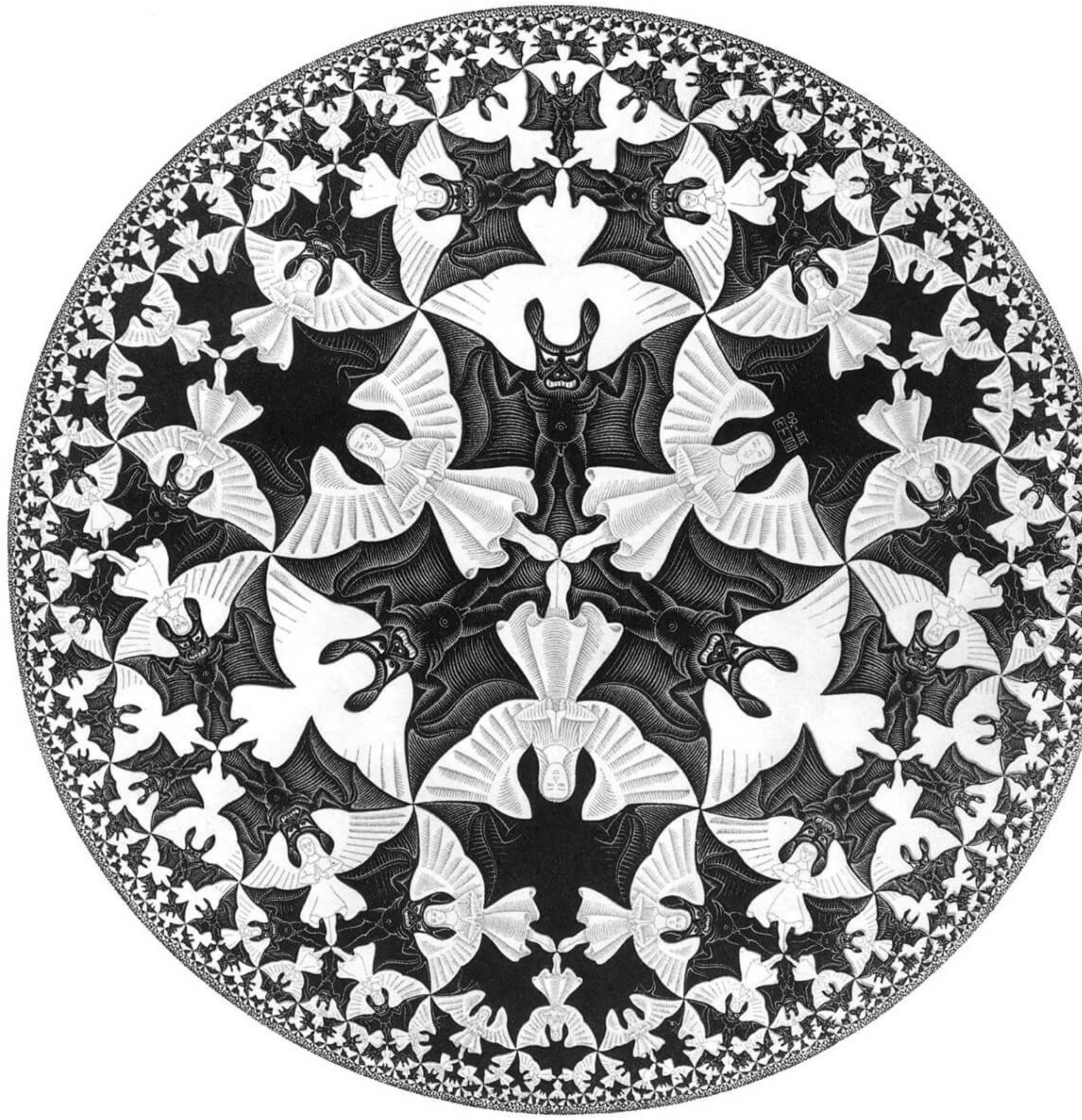
There is an embedding

$$\Delta \hookrightarrow \mathrm{PSL}_2(\mathbb{R})$$

That can be **explicitly given** by square roots, $\sin(\pi/s)$ and $\cos(\pi/s)$ for $s \in \{a, b, c\}$.

There is an action of $\mathrm{PSL}_2(\mathbb{R})$ on the (completed) upper-half complex plane $\mathscr{H}$:

$$\begin{pmatrix} s & t \\ u & v \end{pmatrix} \cdot z = \frac{sz+t}{uz+v}.$$



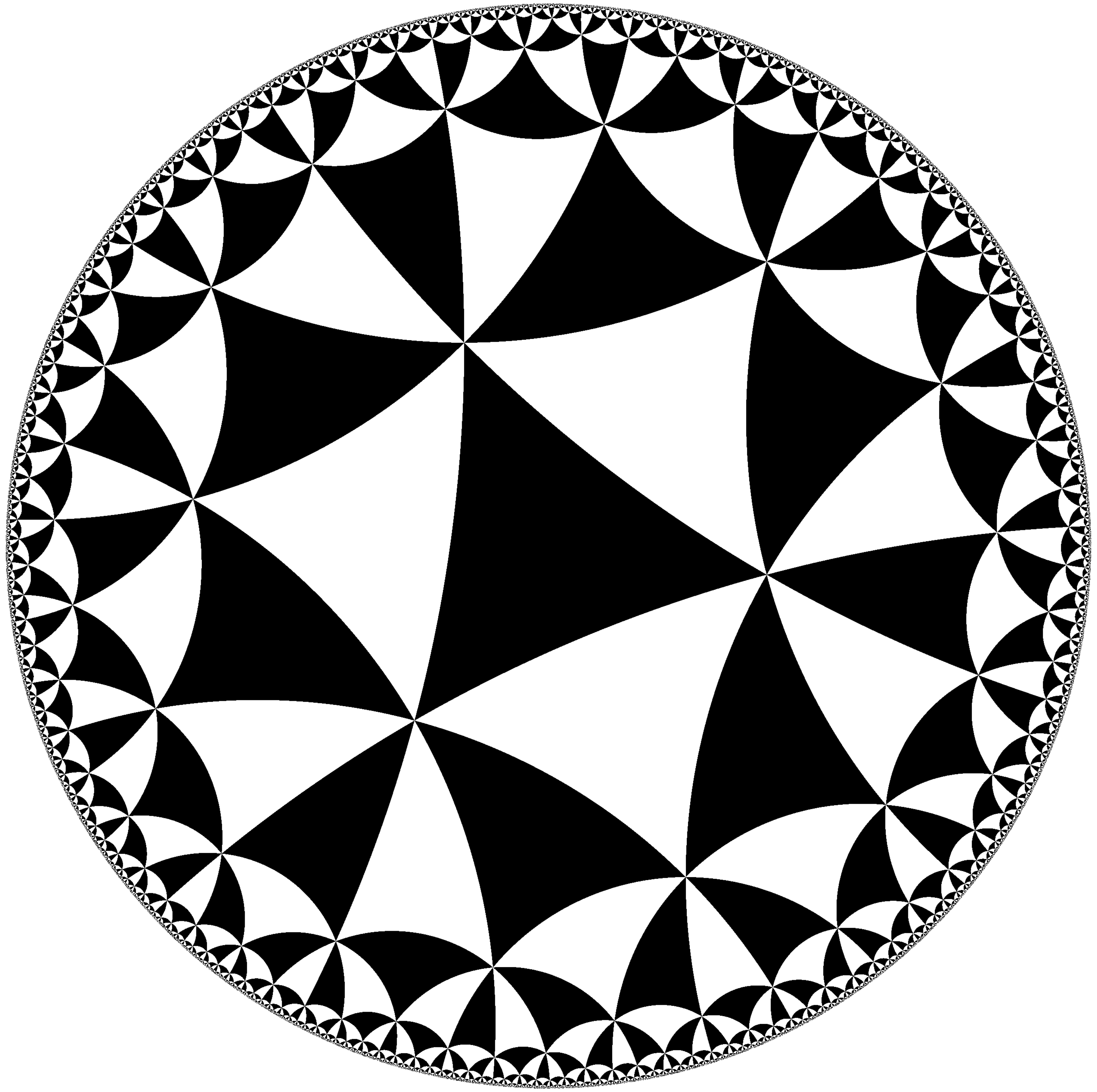
Escher: *Angels and Devils*

Triangular modular curves

Construction

A **triangular modular curve** is an algebraic curve given by the quotient

$$X(1) = X(a, b, c; 1) := \Delta(a, b, c) \backslash \mathscr{H}$$



(4,4,4)

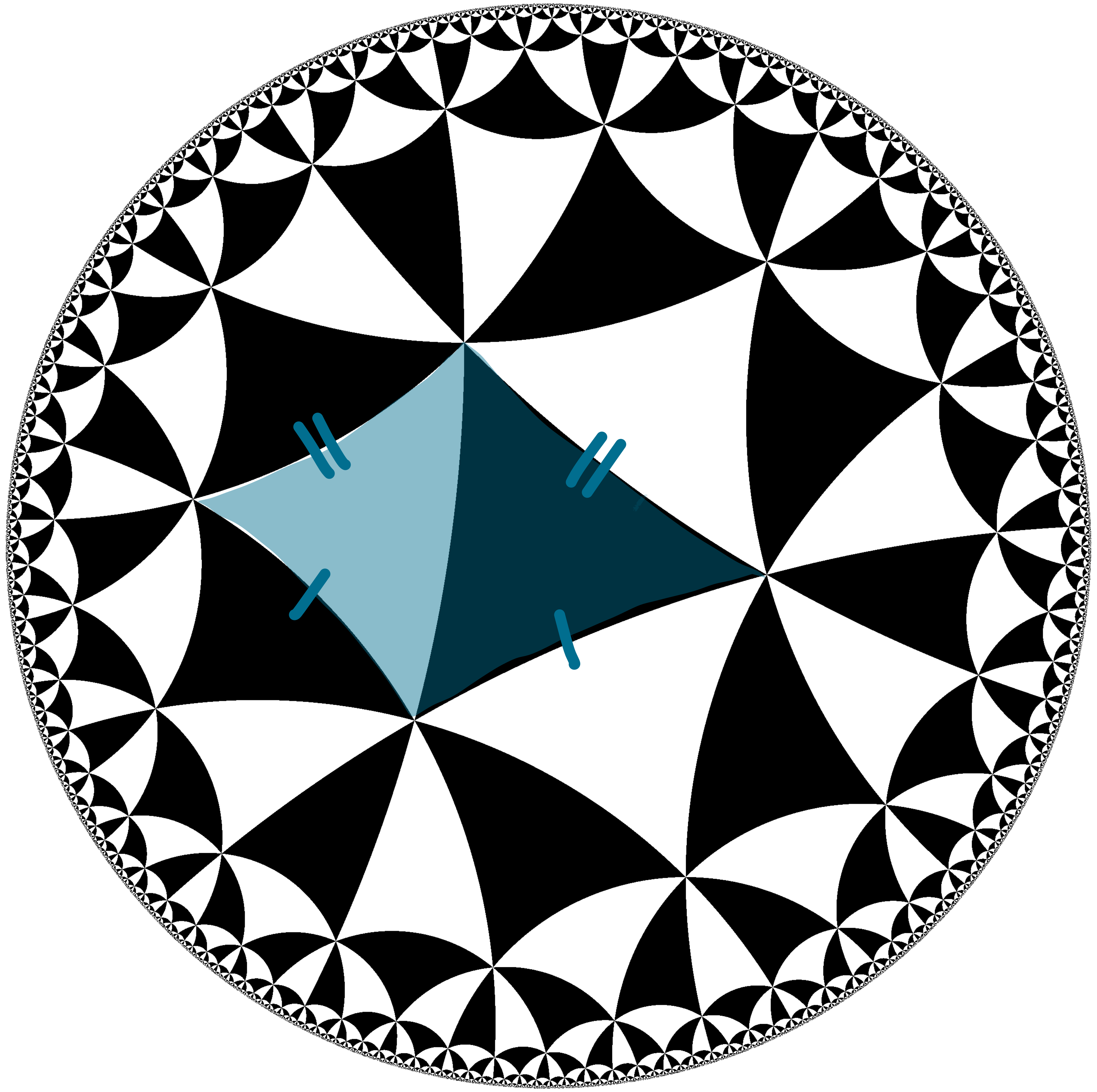
Triangular modular curves

Construction

A **triangular modular curve** is an algebraic curve given by the quotient

$$X(1) = X(a, b, c; 1) := \Delta(a, b, c) \backslash \mathscr{H}$$

By construction $X(a, b, c; 1) \simeq \mathbb{P}^1$, so the curves have genus 0 for all (a, b, c).



(4,4,4)

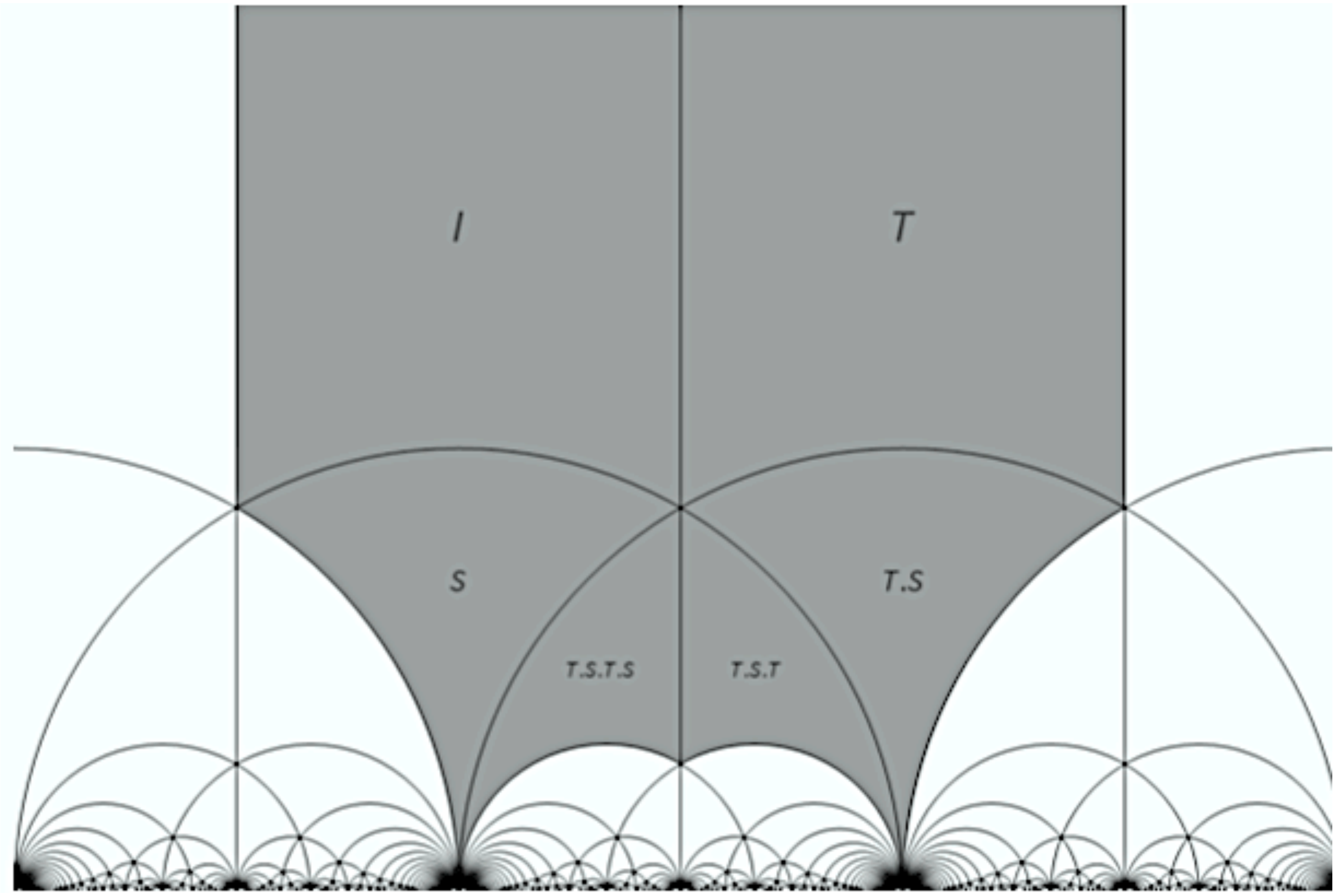
Do we care?

Consider the Legendre family of elliptic curves:

$$E_t : y^2 = x(x-1)(x-t)$$

for a parameter $t \neq 0, 1, \infty$.

- Cyclic covers of $\mathbb{P}^1$ branched at 4 points.
- Parametrized by the modular curve $X(2) = \mathbb{P}^1$.
- We can consider additional level structure. **Example:** specify a cyclic N-isogeny ($X_0(N)$) or an N-torsion point ($X_1(N)$).



Fundamental domain of $\Gamma(2)$. By Paul Kainberger.

Generalizing elliptic curves

Consider the family of curves:

$$X_t : y^m = x^{e_0}(x-1)^{e_1}(x-t)^{e_t}$$

with $t \neq 0, 1, \infty$.

- Cyclic covers of $\mathbb{P}^1$ that are branched at 4 points.
- X_t has a cyclic group of automorphisms of order m defined over $\mathbb{Q}(\zeta_m)$.
- $\mathrm{Prym}(X_t)$ an isogeny factor of $\mathrm{Jac}(\mathrm{X_t})$.

[Cohen & Wolfart '90, Archinard '03] The family $\mathrm{Prym}(X_t)$ extends to a family of abelian varieties over $\mathbb{P}^1$ that are parameterized by triangular modular curves.

Why triangular modular curves?

[Cohen & Wolfart '90, Archinard '03] The family $\mathrm{Prym}(X_t)$ extends to a family of abelian varieties over $\mathbb{P}^1$ that are parameterized by triangular modular curves.

Darmon's program ('04): there is a dictionary between finite index subgroups of the triangle group $\Delta(a, b, c)$ and approaches to solve the generalized Fermat equation

$$x^a + y^b + z^c = 0.$$

Level structure

Let p be a prime with $p \nmid 2abc$. We consider the number field

$$E = E(a,b,c) := \mathbb{Q}\left(\cos\left(\frac{2\pi}{a}\right),\ \cos\left(\frac{2\pi}{b}\right),\ \cos\left(\frac{2\pi}{c}\right),\ \cos\left(\frac{\pi}{a}\right)\cos\left(\frac{\pi}{b}\right)\cos\left(\frac{\pi}{c}\right)\right).$$

Let $\mathfrak{p}/p$ **be a prime of** E**.** There is a surjective homomorphism

$$\pi_{\mathfrak{p}} : \Delta \twoheadrightarrow \mathrm{PXL}_2(\mathbb{Z}_E/\mathfrak{p}).$$

We can decide between PSL_2 and PGL_2 from the behavior of $\mathfrak{p}$ in an extension of E.

Congruence subgroups

$$\pi_{\mathfrak{p}} : \Delta \to \mathrm{PXL}_2(\mathbb{Z}_E/\mathfrak{p}).$$

The **principal congruence subgroup of level $\mathfrak{p}$** is:

$$\Gamma(\mathfrak{p}) := \ker \pi_{\mathfrak{p}} \trianglelefteq \Delta.$$

The **triangular modular curve of level $\mathfrak{p}$** is:

$$X(\mathfrak{p}) = X(a, b, c; \mathfrak{p}) := \Gamma(\mathfrak{p}) \backslash \mathscr{H}$$

These curves come with an associated Belyi map:

$$X(a, b, c; \mathfrak{p}) \to X(a, b, c; 1) \simeq \mathbb{P}^1.$$

Example: $(2,3,7;\mathfrak{p}_7)$

This is the Klein quartic curve!

We understand the cover $X(2,3,7;\mathfrak{p}_7) \to \mathbb{P}^1$:

- The degree is $\#\mathrm{PSL}_2(\mathbb{F}_7) = 168$,
- It is ramified over $0, 1$ and ∞,
- Every ramification point above each $0, 1$ and ∞ has the same degree ($s \in \{a, b, c\}$).

Now we apply this to the Riemann-Hurwitz formula:

$$2g - 2 = -2 \cdot d + \sum_P e_P.$$

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Now we apply this to the Riemann-Hurwitz formula:

$$2g - 2 = -2 \cdot 168 + \frac{168}{2} \cdot (2-1) + \frac{168}{3} \cdot (3-1) + \frac{168}{7} \cdot (7-1).$$

So $g = 3$.

Congruence subgroups

Borel kind

Let $H_0 \leq \mathrm{PXL}_2(\mathbb{Z}_E/\mathfrak{p})$ be the image of the upper triangular matrices in $\mathrm{XL}_2(\mathbb{Z}_E/\mathfrak{p})$.

$$\Gamma_0(\mathfrak{p}) = \Gamma_0(a, b, c; \mathfrak{p}) := \pi_{\mathfrak{p}}^{-1}(H_0).$$

We define the TMC with level $\mathfrak{p}$:

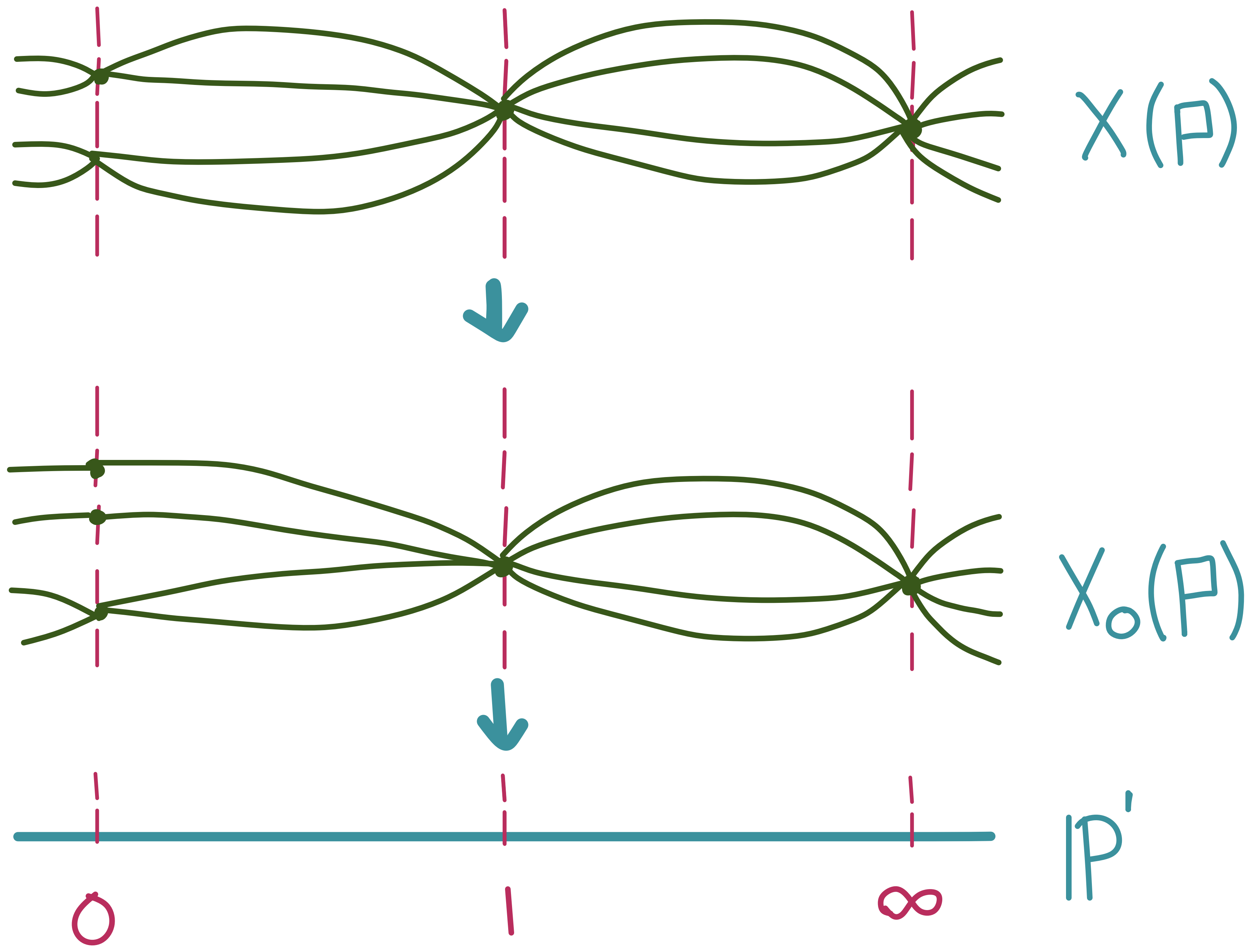
$$X_0(\mathfrak{p}) = X_0(a, b, c; \mathfrak{p}) := \Gamma_0(\mathfrak{p}) \backslash \mathscr{H}.$$

$$X(\mathfrak{p}) \rightarrow X_0(\mathfrak{p}) \rightarrow X(1)$$

The maps to $X(1)$ are Belyi maps!

We can also construct $X_1(a, b, c; \mathfrak{p})$ and we get

$$X(\mathfrak{p}) \rightarrow X_1(\mathfrak{p}) \rightarrow X_0(\mathfrak{p}) \rightarrow X(1)$$

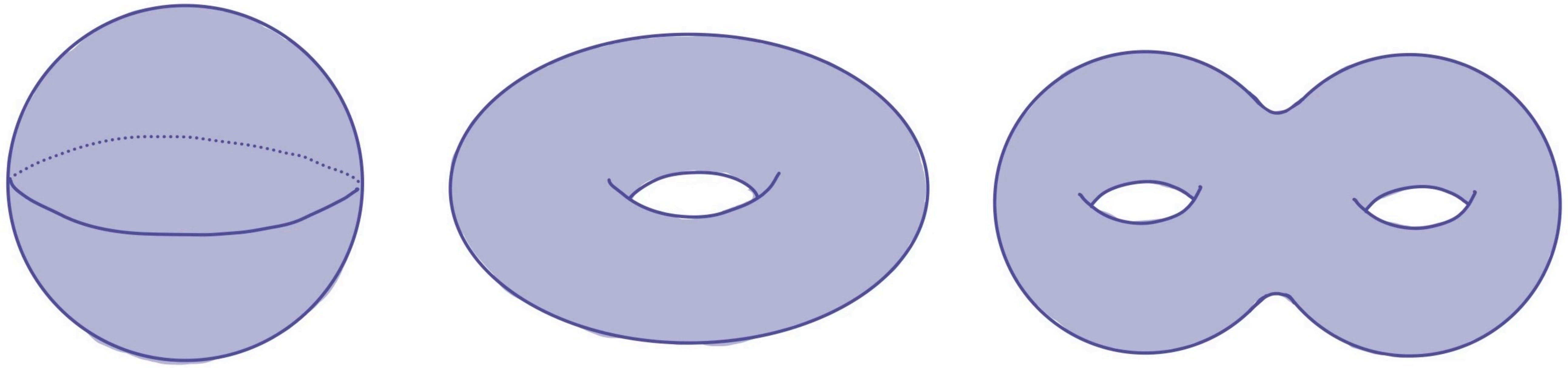


Main theorem

Theorem [DR & Voight '22]

For any $g \in \mathbb{Z}_{\geq 0}$ there are **finitely many** Borel-type triangular modular curves $X_0(a, b, c; \mathfrak{p})$ **of genus** g with nontrivial prime level $\mathfrak{p}$. The number of curves $X_0(a, b, c; \mathfrak{p})$ of genus $g \leq 2$ are as follows:

- 69 curves of genus 0;
- 248 curves of genus 1;
- 453 curves of genus 2.



We have a similar result for $X_1(a, b, c; \mathfrak{p})$

A bound on the number of TMCs of bounded genus

Theorem [DR & Voight '22]. Let $g_0 \geq 0$ be the genus of $X_0(a, b, c; \mathfrak{p})$. Recall that $q := \#\mathbb{F}_{\mathfrak{p}}$. Then

$$q \leq \frac{2(g_0 + 1)}{|-1/42|} + 1.$$

In particular the number of TMCs $X_0(a, b, c; \mathfrak{p})$ of genus g_0 is finite.

We obtain an explicit formula for the genus

$$g(X_0(a, b, c; \mathfrak{p})).$$

Ramification

Lemma. Let $G = \mathrm{PXL}_2(\mathbb{F}_q)$ with $q = p^r$ for p prime. (a, b, c) is a hyperbolic admissible triple. Let $\sigma_s \in G$ have order $s \geq 2$ and if $s = 2$ suppose $p = 2$. Then the action of σ_s on G/H_0 has:

$$\text{orbits of length } s \text{ and } \begin{cases} 0 \text{ fixed points if } s \mid (q+1), \\ 1 \text{ fixed point if } s = p, \\ 2 \text{ fixed points if } s \mid (q-1)\,. \end{cases}$$

In particular s must divide one between $q+1, p, q+1$ for all $s \in \{a, b, c\}$ and we understand the ramification of the cover

$$X_0(\mathfrak{p}) \to \mathbb{P}^1.$$

Enumeration algorithm

Input: $g_0 \in \mathbb{Z}_{\geq 0}$.

Output: A list of $(a, b, c; p)$ such that $X_0(a, b, c; \mathfrak{p})$ has genus bounded by g_0 where $\mathfrak{p}$ is a prime of $E(a, b, c)$ of norm p.

1. Generate a list of possible q values.
2. For each q find all q-admissible hyperbolic triples (a, b, c).
3. Compute the genus g of $X_0(a, b, c; \mathfrak{p})$ by checking divisibility.
4. If $g \leq g_0$ add $(a, b, c; p)$ to the list lowGenus.

Future work

Compute explicit formulas for composite level.

Find models using Belyi maps and compute rational points of TMCs of low genus. [Klug, Musty, Schiavone & Voight, '14].

Example: the curve $X_0(3,3,4;\mathfrak{p}_7)$ is defined over the number field k with defining polynomial $x^4 - 2x^3 + x^2 - 2x + 1$. We have

$$X_0(3,3,4;\mathfrak{p}_7) \simeq \mathbb{P}^1_k.$$

Conjecture. For all $g \in \mathbb{Z}_{\geq 0}$, there are only finitely many admissible triangular modular curves of genus g of nontrivial level $\mathfrak{N} \neq (1)$ with $\Delta(a,b,c)$ maximal.

Output for $X_0(a, b, c; p)$ of genus 0

a	b	c	p
2	3	7	7
2	3	7	2
2	3	7	13
2	3	7	29
2	3	7	43
2	3	8	7
2	3	8	3
2	3	8	17
2	3	8	5
2	3	9	19
2	3	9	37
2	3	10	11
2	3	10	31
2	3	12	13
2	3	12	5
2	3	13	13
2	3	15	2
2	3	18	19
2	4	5	5
2	4	5	3
2	4	5	11
2	4	5	41
2	4	6	5
2	4	6	7
2	4	6	13
2	4	8	3
2	4	8	17
2	4	12	13
2	5	5	5
2	5	5	11
2	5	10	11
2	6	6	7
2	6	6	13
2	6	7	7
2	8	8	3
3	3	4	7
3	3	4	3
3	3	4	5
3	3	5	2
3	3	6	13
3	3	7	7
3	4	4	5
3	4	4	13
3	6	6	7
4	4	4	3
4	4	5	5
2	3	∞	2
2	3	∞	3
2	3	∞	5
2	4	∞	3
2	∞	∞	3
3	3	∞	3
3	∞	∞	2
3	∞	∞	3
∞	∞	∞	3

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