Basic Constructions

(Milne, Chapter 5)

Outline:

- Products and Fibred Products
- Limits
- Extension of scalars
- Restriction of scalars
- Transporters (Group Actions)
- Galois Descent (?)

Note: k is a commutative ring

Recall: An affine group corresponds to a functor $G : \mathrm{Alg}_k \longrightarrow \mathrm{Grp}$ such that the associated forgetful functor $\mathrm{Alg}_k \longrightarrow \mathrm{Set}$ is representable.

Products

Let $(G_i)_{i\in I}$ be a family of affine groups over k.

The product of $(G_i)_{i\in I}$ is the functor G:

$$R \rightsquigarrow \prod_{i\in I} G_i(R),$$

represented by $\bigotimes_{i\in I} \mathcal{O}(G_i)$:

sets

$$G(R) \simeq \prod_{i\in I} \mathrm{Hom}(\mathcal{O}(G_i), R)$$
$$\simeq \mathrm{Hom}\left(\bigotimes_{i\in I} \mathcal{O}(G_i), R\right)$$

⚠ Infinite products of affine algebraic groups do not exist in general.

closed subgroup of GL_n

Fibred Products

Let G_1, G_2 and H be affine groups

Let $$G_1 \longrightarrow H \longleftarrow G_2$$

be homomorphisms of affine groups.

The **fibred product** $G_1 \times_H G_2$ is the functor:

$$R \rightsquigarrow G_1(R) \times_{H(R)} G_2(R).$$

This functor is represented by

$$\mathcal{O}(G_1) \otimes_{\mathcal{O}(H)} \mathcal{O}(G_2).$$

Special case: Let $\alpha, \beta : G \longrightarrow H$ be hom. of affine grps. Then

$$Eq(\alpha, \beta) := G \times_{\alpha, H, \beta} G$$

is the **equilizer** of α and β.

Special case: $\alpha : G \longrightarrow H$ hom. of affine grps.

$$Ker(\alpha) := Eq(\alpha, e) = G \times_H *$$

is the **kernel** of α.

Coordinate ring?

$$\mathrm{Hom}_{\mathcal{O}(H)\text{-alg.}}\left(\mathcal{O}(G_1) \times_{\mathcal{O}(H)} \mathcal{O}(G_2), R\right)$$

$$\text{is}$$

$$\mathrm{Hom}_{\mathcal{O}(H)\text{-alg}}\left(\mathcal{O}(G_1), R\right) \times \mathrm{Hom}_{\mathcal{O}(H)\text{-alg}}\left(\mathcal{O}(G_2), R\right)$$

$$\mathrm{Hom}_{k\text{-alg}}\left(\mathcal{O}(G_1) \times_{\mathcal{O}(H)} \mathcal{O}(G_2), R\right)$$

$$\text{is}$$

$$\mathrm{Hom}_{k\text{-alg}}\left(\mathcal{O}(G_1), R\right) \times_{\mathrm{Hom}_{k\text{-alg}}(\mathcal{O}(H), R)} \mathrm{Hom}_{k\text{-alg}}\left(\mathcal{O}(G_2), R\right)$$

Limits

> **Theorem: (3.1)** Let F be a functor from a small category I to the category of affine groups over k. Then the functor
>
> $$R \rightsquigarrow \varprojlim F(R)$$
>
> is an affine group, and it is the inverse limit of F in the category of affine groups.

idea of the proof:

$$\underleftarrow{F} : R \rightsquigarrow \boxed{\varprojlim F(R)}$$

This is the inverse limit of $i \rightsquigarrow F_i(R)$
$I \longrightarrow Grp$

① $\underleftarrow{F}$ is an affine group

$$\prod_{i \in obj(I)} F_i \overset{f}{\underset{g}{\rightrightarrows}} \prod_{u \in arr(I)} F_{target(u)}$$

$$p_u f := p_{target(u)} \qquad p_u g = F_u \circ p_{dom(u)}$$

$$p_u : \prod_{u \in arr(I)} F_{target(u)} \longrightarrow F_{target(u)}$$

② $\underleftarrow{F} = \varprojlim F$

Extension of Scalars

k' is a k-algebra.

If R is a k'-algebra, then it is a k-algebra:

$$k \to k' \longrightarrow R$$

$$\boxed{\begin{array}{l}\text{If } G : \mathrm{Alg}_k \longrightarrow \mathrm{Grp} \text{ is an affine } k\text{-group, then the functor:} \\ \qquad\qquad G_{k'} : R \rightsquigarrow G(R) \\ \text{is the extension of scalars of } G.\end{array}}$$

Coordinate ring?

$$\mathrm{Hom}_{k'\text{-alg}}(k' \otimes A, R) \simeq \mathrm{Hom}_{k\text{-alg}}(A, R)$$

Then $\mathcal{O}(G_{k'}) = k' \otimes \mathcal{O}(G)$

Example: V projective f.g k-mod

W k'-algebra

$$\star \quad D_a(V) : \underset{\substack{\uparrow \\ k\text{-alg}}}{R} \rightsquigarrow (\mathrm{Hom}_{k\text{-lin}}(V, R), +)$$

$$\mathrm{Hom}_{k\text{-alg}}(\mathrm{Sym}(V), R) \simeq \mathrm{Hom}_{k\text{-lin}}(V, R)$$

$\star$ k-linear map $\quad V \longrightarrow W'$

$\Big\downarrow$ extend

k'-linear map $\quad V \otimes k' \longrightarrow W'$

$$\begin{array}{ccc} \mathbb{D}a(V)_{k'} : R & \rightsquigarrow & \mathrm{Hom}_{k\text{-lin}}(V, R) \\ \| & & \wr\!\!\| \\ \mathbb{D}a(V_{k'}) : R & \rightsquigarrow & \mathrm{Hom}_{k'\text{-lin}}(V \otimes k', R) \end{array}$$

Restriction of the base ring (Weil Restriction)

k' is a k-algebra that is f.g. and projective as a k-mod.

If $G: \mathrm{Alg}_{k'} \to \mathrm{Grp}$ is an affine k'-group, then the functor:

$$(G)_{k'/k} : R \rightsquigarrow G(k' \otimes_k R)$$

is the restriction of the base ring of G.

Notation: $(G)_{k'/k} = \mathrm{Res}_{k'/k} = \prod_{k'/k}$

Proposition: If G is an affine k'-group, then
(Miracle) $(G)_{k'/k}$ is an affine k-group and

for all affine k-groups H and all affine k'-groups G, there are canonical isomorphisms

$$\mathrm{Hom}_k(H, (G)_{k'/k}) \simeq \mathrm{Hom}_{k'}(H_{k'}, G)$$

that are natural in both H and G.

Corollary: $G \rightsquigarrow (G)_{k'/k}$ is right adgoint to $G_{k'}$

Properties

① Right adjoint $\Longrightarrow$

products	$\rightsquigarrow$	products
fibred prod.	$\rightsquigarrow$	fibred prod.
equilizers	$\rightsquigarrow$	equilizers
kernels	$\rightsquigarrow$	kernels.

② $k \longrightarrow k' \longrightarrow k''$ homomorphisms of rings such that

k' is finitely generated and projective over k

k'' is finitely generated and projective over k'

Then,

$$\mathrm{Res}_{k'/k} \circ \mathrm{Res}_{k''/k'} \simeq \mathrm{Res}_{k''/k}$$

③ k' is a k-algebra

K is a k-algebra

$$\left(\mathrm{Res}_{k'/k} G\right)_K \simeq \mathrm{Res}_{(k' \otimes_k K)/K}(G_K)$$

④ $k' = k_1 \times \cdots \times k_n$, $\quad k_i$ is a k-algebra that is f.g and proj. as a k-mod.

$$(G)_{k'/k} \simeq (G)_{k_1/k} \times \cdots \times (G)_{k_n/k}.$$

⑤ k field, k' finite separable ext. of k.
K field containing all k-conj. of k'
$(|\mathrm{Hom}_k(k', K)| = [k', k])$

Then $$\left(\mathrm{Res}_{k'/k} G\right)_K \simeq \prod_{\alpha : k' \to K} \alpha G$$

affine grp that is the extension of scalars w.r.t. $\alpha : k' \to K$

Actions and Transporters

Let G be an affine monoid over k.
Let X be a functor : $\mathrm{Alg}_k \longrightarrow \mathrm{Set}$

An action of G on X is a natural transformation $G \times X \longrightarrow X$ such that $G(R) \times X(R) \longrightarrow X(R)$ is an action of $G(R)$ on $X(R)$ for all k-algebras R.

Y, Z subfunctors of X.

The transporter of Y into Z is the functor

$$T_G(Y,Z) : R \rightsquigarrow \{ g \in G(R) \mid \underbrace{gY \subseteq Z}_{gY(R') \subseteq Z(R') \ \forall R\text{-alg. } R'} \}$$

Is the transporter an affine group?

Not in general.

We need:
- Y representable by a k-alg. free as k-mod
- Z closed in X

Let Z be a subfunctor of a functor $Y: \mathrm{Alg}_k \to \mathrm{Set}$. We say that Z is closed if, for every k-alg. A and natural transformation $h^A \to Y$, the fibred product $Z \times_Y h^A$ is represented by a quotient of A.

Or...

Z is closed in Y if and only if, for every k-alg A and $\alpha \in Y(A)$, the functor (of k-alg)

$$R \rightsquigarrow \{\varphi : A \to R \mid \varphi(\alpha) \in Z(A)\}$$

is represented by a quotient of A.

Example: Y is the functor $\mathbb{A}^n = (R \rightsquigarrow R^n)$. Then a subfunctor is closed iff it is defined by a finite set of polynomials in $k[X_1, \ldots, X_n]$

Galois descent of affine groups

k field

Ω/k Galois ext.

$\Gamma = \mathrm{Gal}(\Omega/k)$

Prop. The functor $G \rightsquigarrow G_\Omega$ from affine groups over k to affine groups over Ω equipped with a continuous action of Γ is an equivalence of categories.

Example. $G = \mathbb{G}_m \qquad k = \mathbb{R} \qquad k' = \mathbb{C}$

Extension: $\mathbb{G}_m : \mathrm{Alg}_{\mathbb{R}} \longrightarrow \mathrm{Grp}$

$$\mathbb{G}_{m_{\mathbb{C}}} : \mathrm{Alg}_{\mathbb{C}} \longrightarrow \mathrm{Grp}$$

$$A \rightsquigarrow \mathbb{G}_m(A) = A^\times$$

Restriction: $\underset{\substack{\uparrow \\ \mathbb{R}\text{-alg}}}{(\mathbb{G}_m)_{\mathbb{C}/\mathbb{R}}(A)} \rightsquigarrow \mathbb{G}_m(\mathbb{C} \otimes_{\mathbb{R}} A)$

$$\mathbb{G}_m(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}) = \mathbb{C}^\times$$

$$\mathbb{G}_m(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}) = \mathbb{C}^\times \times \mathbb{C}^\times$$

$$\left((\mathbb{G}_m)_{\mathbb{C}/\mathbb{R}}\right)_{\mathbb{C}} = \mathrm{Res}_{\mathbb{C}\otimes_{\mathbb{R}}\mathbb{C}/\mathbb{C}}(\mathbb{G}_{m_{\mathbb{C}}})$$

$$\mathrm{Res}_{\mathbb{C}/\mathbb{C}}(\mathbb{G}_{m_{\mathbb{C}}}) \times \mathrm{Res}_{\mathbb{C}/\mathbb{C}}(\mathbb{G}_{m_{\mathbb{C}}})$$

$$\mathbb{G}_{m_{\mathbb{C}}} \times \mathbb{G}_{m_{\mathbb{C}}}$$