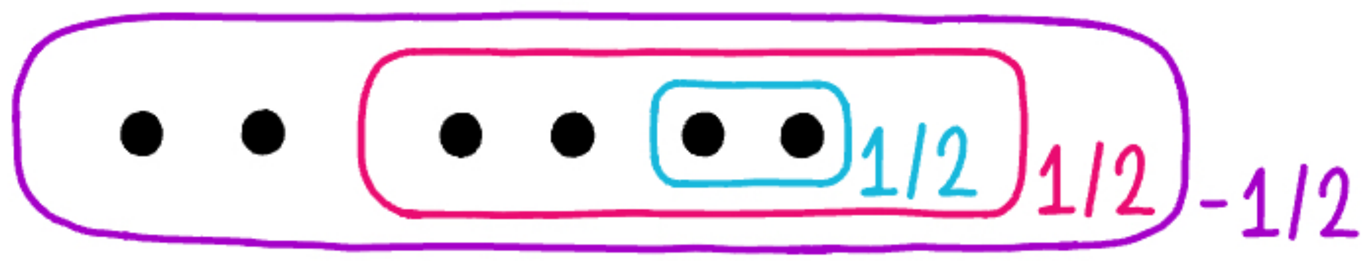
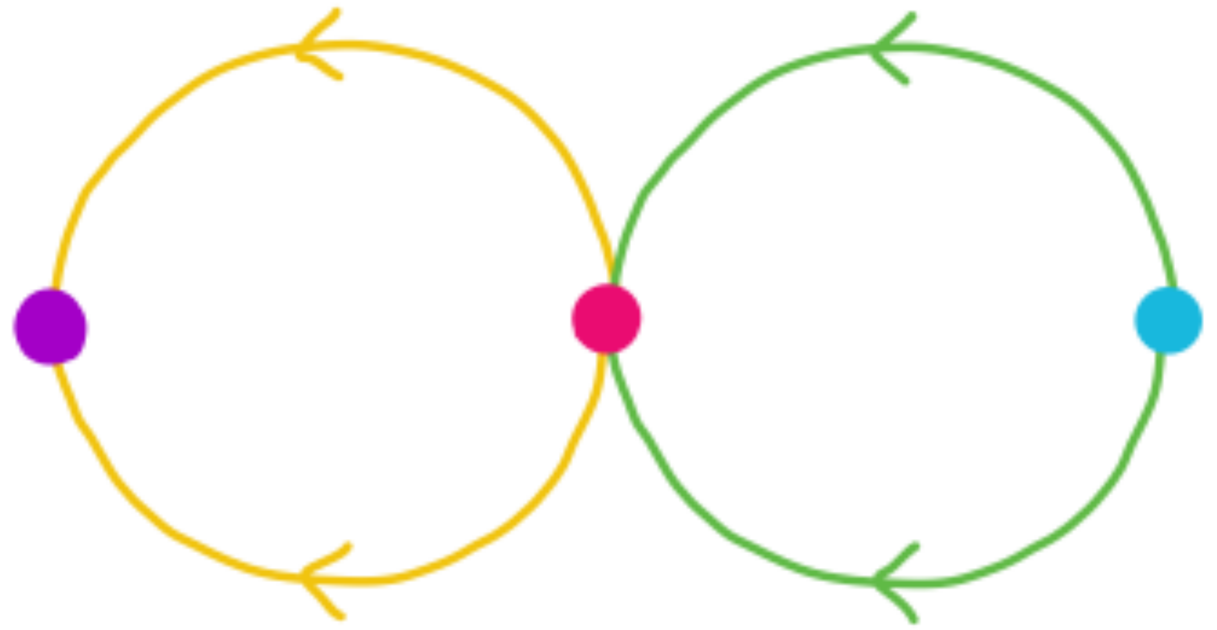
Local heights away from p for quadratic Chabauty

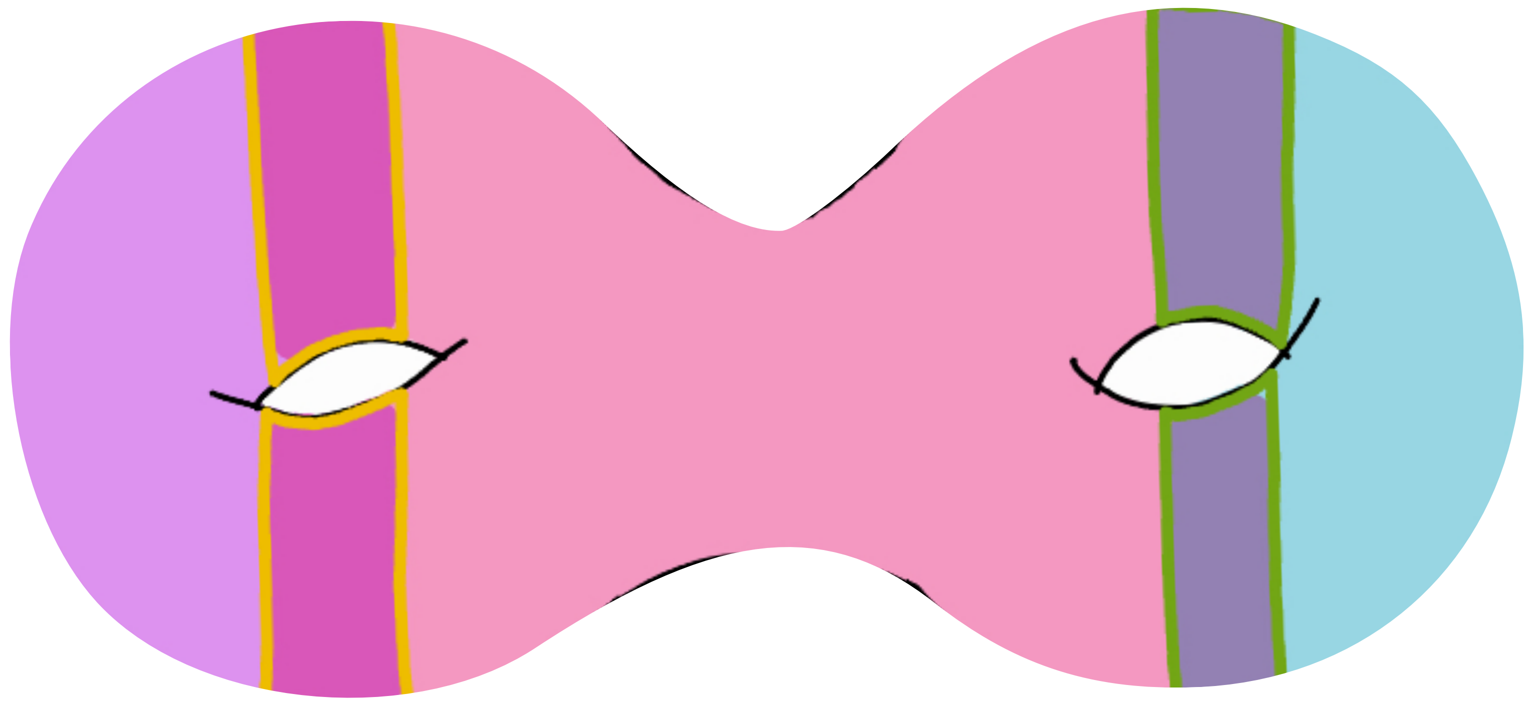
Juanita Duque-Rosero

Boston University ⟶ University of Groningen

Joint work with Alexander Betts, Sachi Hashimoto, and Pim Spelier.







Quadratic Chabauty

Theorem [Balakrishnan—Dogra, '18]. Let C be a nice curve of genus g and J be its Jacobian with Mordell-Weil rank r. Suppose that $r = g$ $(+\epsilon)$, and that $Z \subset C \times C$ is a trace 0 correspondence fixed by the Rosati involution. Then there exists a quadratic function

$$\eta_Z \colon \mathrm{Lie}\left(J_{\mathbb{Q}_p}\right) \to \mathbb{Q}_p$$

for which $C(\mathbb{Q})$ is contained in the locus inside $C(\mathbb{Q}_p)$ cut out by the equation

$$\eta_Z(\log(z)) - h_{Z,p}(z) \in \Omega,$$

where $\Omega = \left\{ \sum_{\ell \neq p} h_{Z,\ell}(x_\ell) \;:\; x_\ell \in C(\mathbb{Q}_\ell) \right\}.$

Quadratic Chabauty and heights

Key input: Let p be a prime number and $Z \subset C \times C$ be a trace 0 correspondence fixed by the Rosati involution. There the associated p-**adic** (Coleman—Gross) **height function** $h_Z : C(\mathbb{Q}) \to \mathbb{Q}_p$ can be decomposed as

$$h_Z(Q) = \sum_{\ell} h_{Z,\ell}(Q),$$

where $h_{Z,\ell} : C(\mathbb{Q}_\ell) \to \mathbb{Q}_p$.

- For $\ell \neq p$ the height function $h_{Z,\ell}(x_\ell)$ takes only finitely many values for $x_\ell \in C(\mathbb{Q}_\ell)$.
- If $\ell \neq p$ is a prime of potential good reduction, then $h_{Z,\ell}(x_\ell) = 0$ for all $x_\ell \in C(\mathbb{Q}_\ell)$.

Today's problem: **to compute heights $h_{Z,\ell}$ for $\ell \neq p$ odd of bad reduction.**

Quadratic Chabauty and heights

Many of the applications of quadratic Chabauty so far concern curves

- with trivial local height contributions away from p; or
- whose special fibre at primes of bad reduction consists of a unique irreducible component; or
- for which one can use elliptic curve factors.

Today's problem: **to compute heights $h_{Z,\ell}$ for $\ell \neq p$ odd of bad reduction**.

Theorem [Betts—D.—Hashimoto—Spelier, '25]. Let $C/\mathbb{Q}$ be a hyperelliptic curve that admits a model $y^2 = f(x)$, where $f(x)$ is separable and of degree ≥ 3. Let $Z \subset C \times C$ be a trace 0 correspondence fixed by the Rosati involution. Let p be a prime number. Then there is a provably correct algorithm to compute the function $h_{Z,\ell}$ for all odd primes of bad reduction $\ell \neq p$.

Explicit height computations: how?

Theorem [Betts—Dogra, ’20]. Let p be a prime number and $Z \subset C \times C$ be a trace 0 correspondence fixed by the Rosati involution. Then, there is an explicit formula for computing $h_{Z,\ell}$ in terms of:

- the induced action of Z_* on the homology $\mathrm{H}_1(\Gamma, \mathbb{Q})$ of the dual graph Γ of the geometric special fibre, and
- the traces $\mathrm{tr}_v(Z)$ attached to vertices of Γ.

This formula uses a semistable model of C.

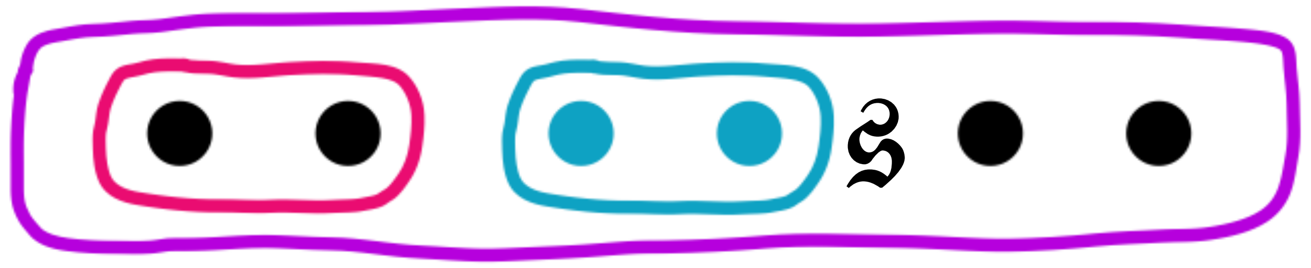
Explicit height computations: how?

We need to understand:

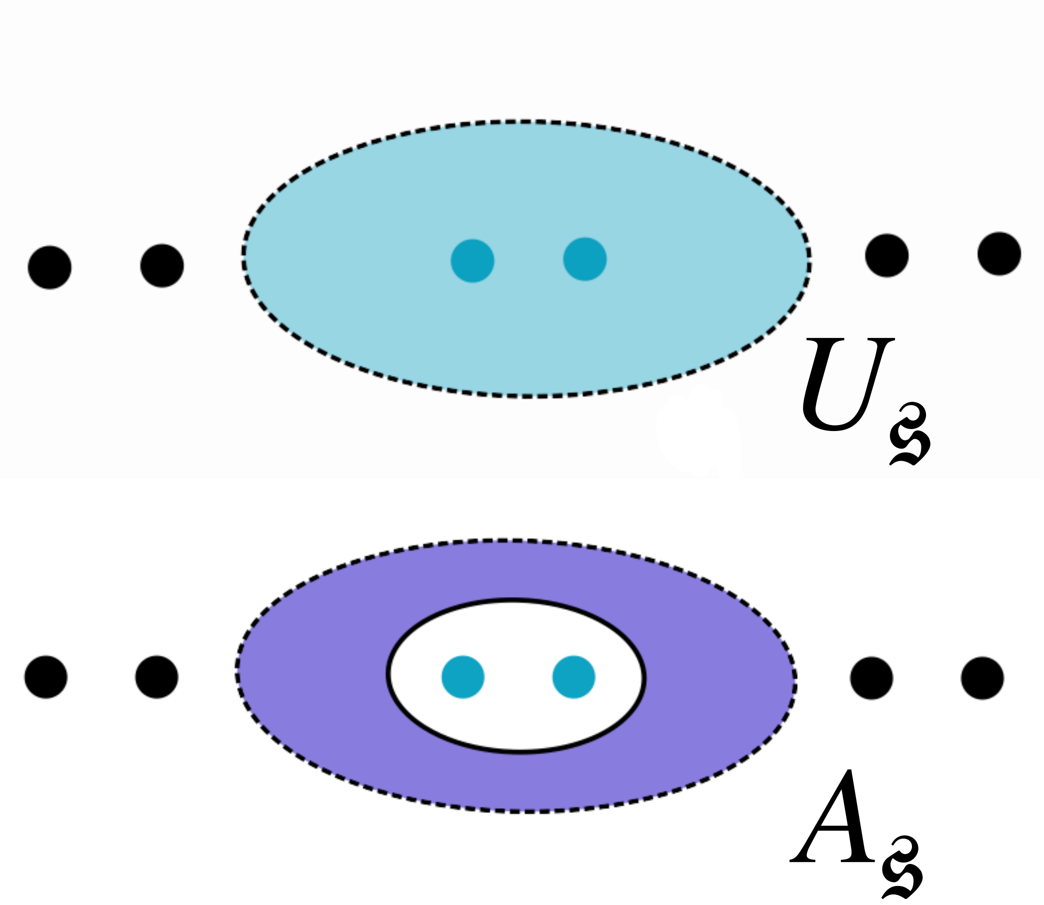
1. A semistable model of C.
2. The induced action of Z_* on the homology $\mathrm{H}_1(\Gamma, \mathbb{Q})$ of the dual graph Γ of the geometric special fibre.
3. The traces $\mathrm{tr}_v(Z)$ attached to vertices of Γ.

1. A semistable covering

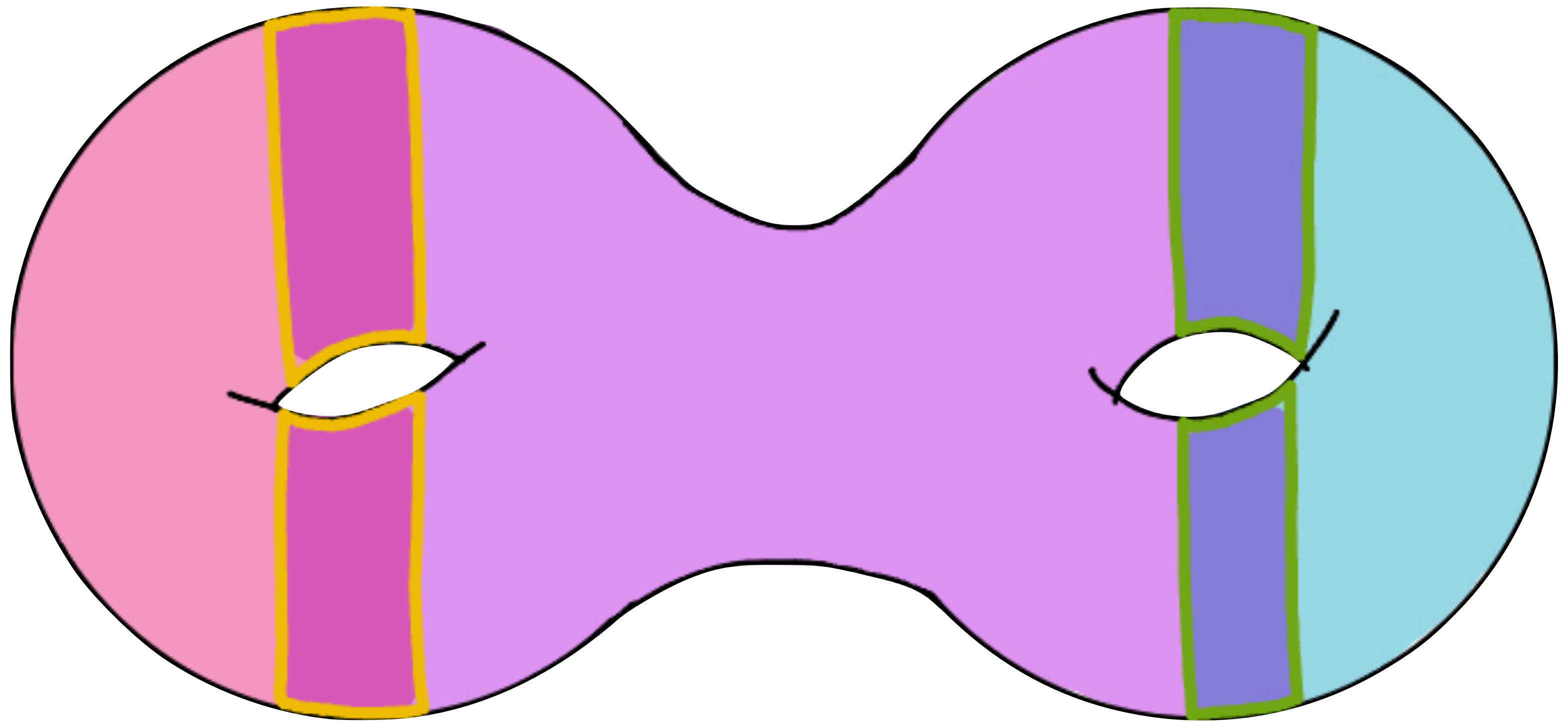
Instead of finding a semistable model of C (or work explicitly with them), we construct a semistable covering of C using cluster pictures [DDMM '23].



Cluster picture: roots of $f(x)$.



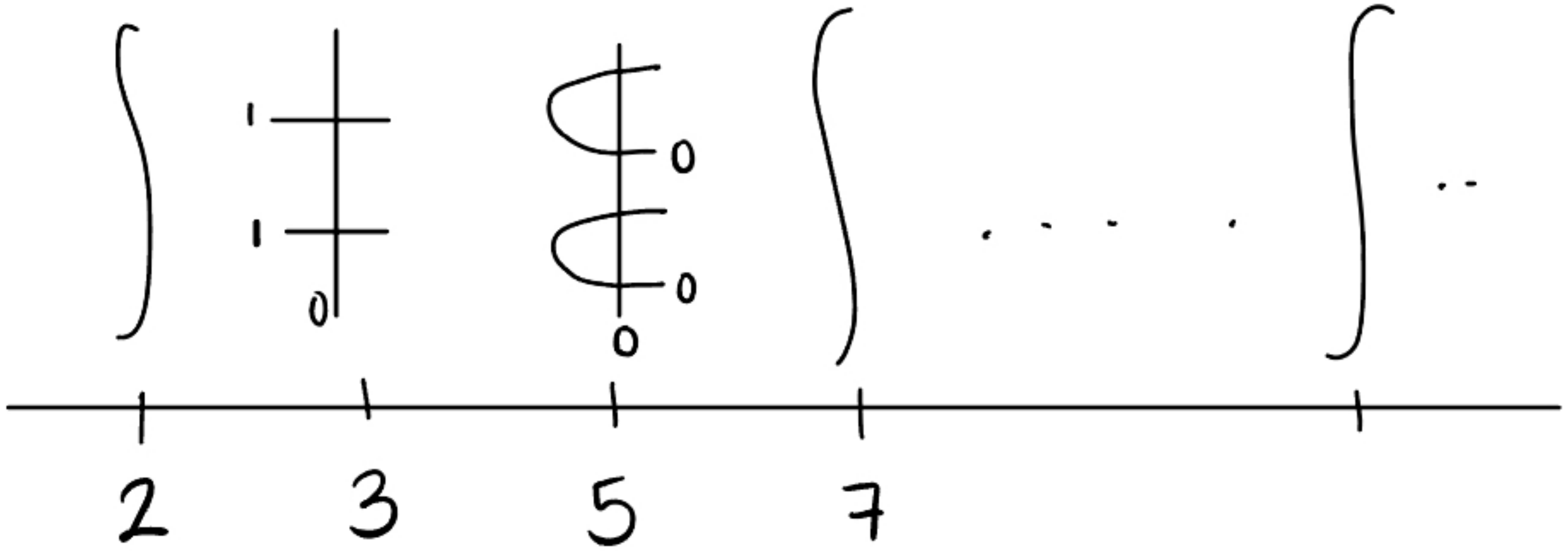
Semistable covering of $\mathbb{P}^{1,\mathrm{an}}$



Semistable covering of C^{an}

Example

$$C : y^2 = x^6 + \frac{18}{5}x^4 + \frac{6}{5}x^3 + \frac{9}{5}x^2 + \frac{6}{5}x + \frac{1}{5}$$



Example

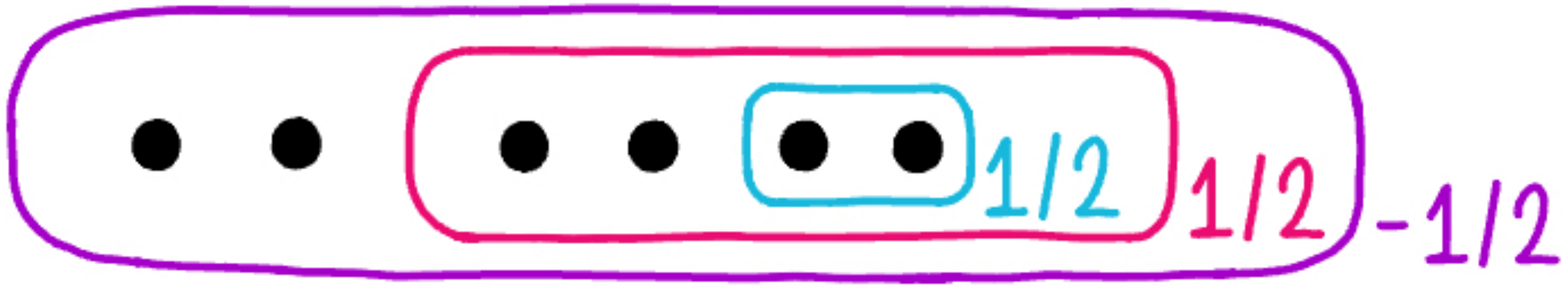
$$C : y^2 = x^6 + \frac{18}{5}x^4 + \frac{6}{5}x^3 + \frac{9}{5}x^2 + \frac{6}{5}x + \frac{1}{5}, \qquad p = 5$$

```
> R<x> := PolynomialRing(pAdicField(5));
> f:= x^6 + 18/5*x^4 + 6/5*x^3 + 9/5*x^2 + 6/5*x + 1/5;
> Factorization(f);
[
    <(5 + O(5^21))*x + 2068953077662*5 + O(5^20), 1>,
    <(5^2 + O(5^22))*x^2 + (6373813938821*5^2 + O(5^21))*x + 343719541919*5^2 +
        O(5^20), 1>,
    <(5 + O(5^21))*x - 1185825657571*5 + O(5^20), 1>,
    <(5^2 + O(5^22))*x^2 - (7256941358912*5^2 + O(5^21))*x - 3436100256277*5 +
        O(5^20), 1>
]
```

Example

$$C : y^2 = x^6 + \frac{18}{5}x^4 + \frac{6}{5}x^3 + \frac{9}{5}x^2 + \frac{6}{5}x + \frac{1}{5}, \qquad p = 5$$

```
> fL := SemistableField(f, p : prec := 30, printing := false);
> roots := Roots(fL);
> [Valuation(roots[1][1]-roots[i][1]) : i in [2..6]];
[ 0, 0, 0, -2, -2 ]
> [Valuation(roots[2][1]-roots[i][1]) : i in [1..6]|i ne 2];
[ 0, 2, 0, -2, -2 ]
```



Example

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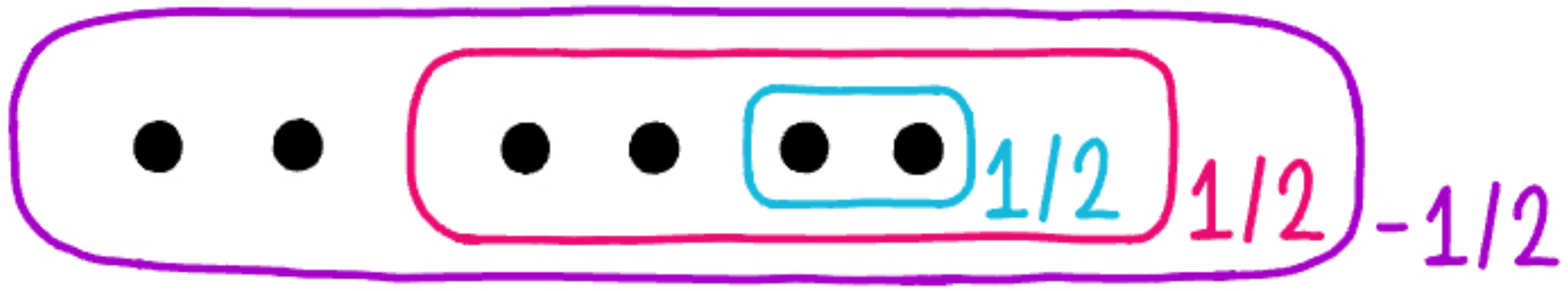
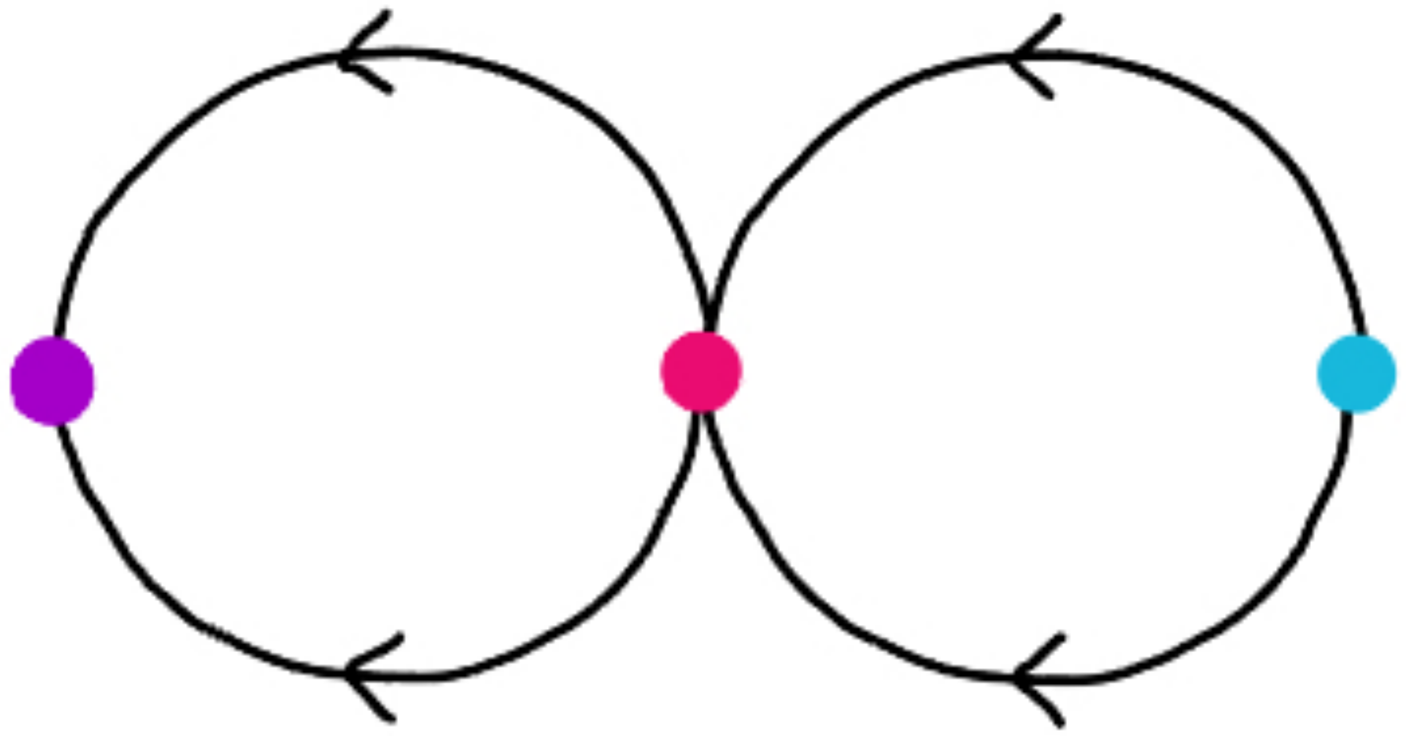
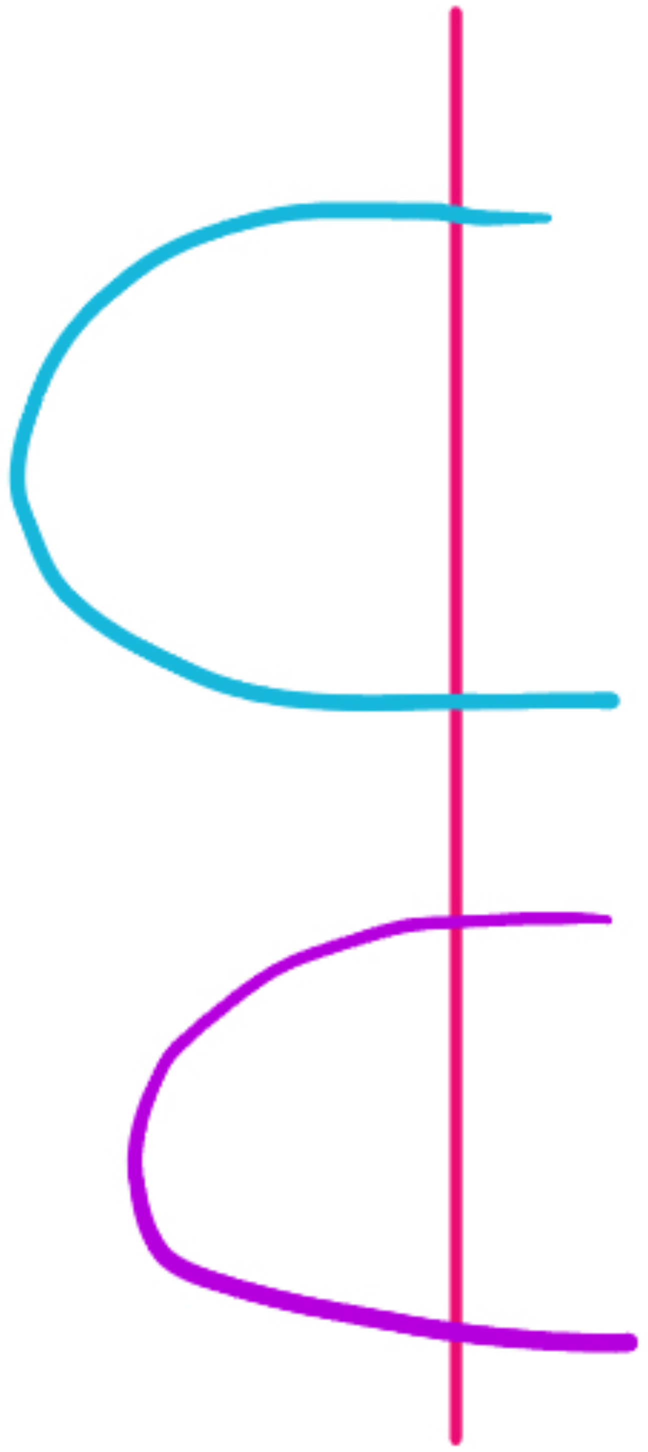
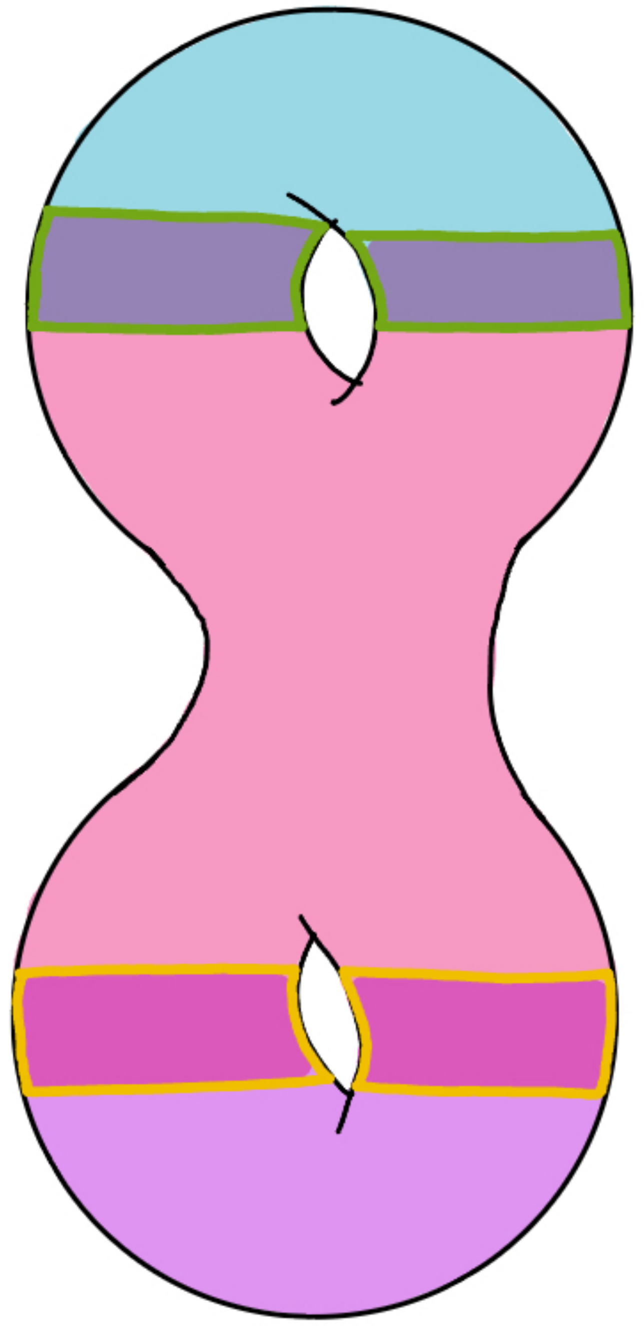








2. The action of Z_* on $\mathrm{H}_1(\Gamma, \mathbb{Q})$: Coleman - Iovita

We represent Z as a divisor in $C \times C$ [Costa—Mascot—Sijsling—Voight, '19].

This allows us to understand the action of Z_* on $\mathrm{H}^0(C_{\mathbb{Q}_\ell}, \Omega^1_C)$.

Theorem [Coleman & Iovita, '99]. The map

$$\mathrm{H}^0(C_{\mathbb{Q}_\ell}, \Omega^1_C) \to \mathrm{H}_1(\Gamma, \mathbb{Q}_\ell), \text{ given by } \omega \mapsto \sum_{e \in E(\Gamma)} \mathrm{Res}_{A_{\vec{e}}}(\omega) \cdot \vec{e},$$

is surjective. Moreover, the map is an isomorphism if every component of the ℓ-adic special fibre has genus 0.

We can compute the action of Z_* on $\mathrm{H}_1(\Gamma, \mathbb{Q}_\ell)$ up to any desired ℓ-adic precision.

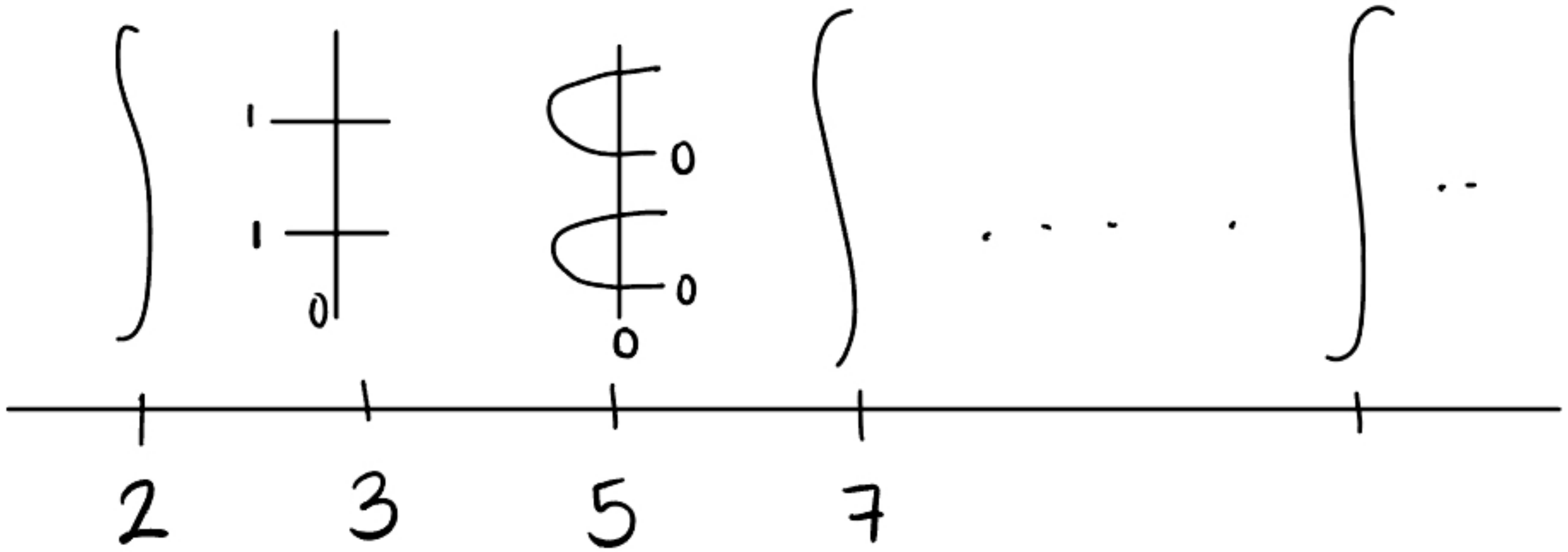
Coleman - Iovita done explicitly

Theorem [Betts—D—Hashimoto—Spelier, '25]. The endomorphism of $\mathrm{H}_1(\Gamma, \mathbb{Q})$ given by Z_* is defined over $\mathbb{Z}$, and has operator norm $\leq \sqrt{d_1 d_2}$, where d_1 and d_2 are the degrees of the two projections $Z \to C$.

Finite precision is enough!

Local heights computations: an example

$$C : y^2 = x^6 + \frac{18}{5}x^4 + \frac{6}{5}x^3 + \frac{9}{5}x^2 + \frac{6}{5}x + \frac{1}{5}$$



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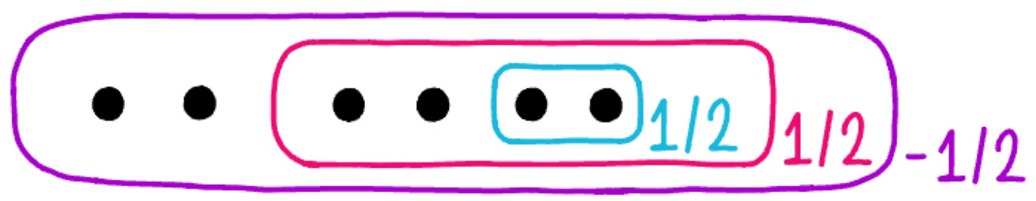
Pick the correspondence Z from the endomorphism $\sqrt{13}$ on the Jacobian. Let $\ell = 5$.



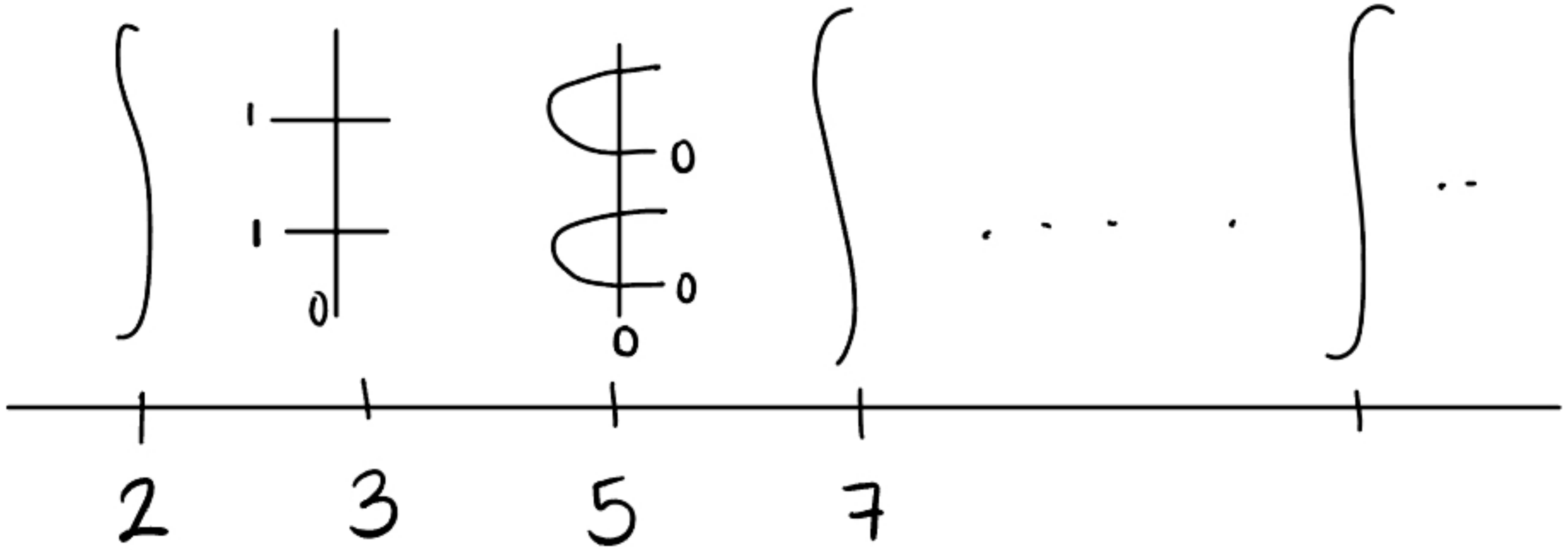
Cluster picture

Berkovich space decomposition

$$h_{Z,5}(x : y : z) = \begin{cases} +\frac{3}{4}\log_p(5) & \text{if } z \equiv 0 \bmod 5, \\ -\frac{3}{4}\log_p(5) & \text{if } x/z \equiv 2 \bmod 5, \\ 0 & \text{otherwise.} \end{cases}$$

Local heights computations: an example

$$C : y^2 = x^6 + \frac{18}{5}x^4 + \frac{6}{5}x^3 + \frac{9}{5}x^2 + \frac{6}{5}x + \frac{1}{5}$$



$$h_{Z,3}(x:y:z) = \begin{cases} 0 & \text{if } z \equiv 0 \bmod 3, \\ -\frac{2}{3}\log_p(3) & \text{if } x/z \equiv 1 \bmod 3. \end{cases}$$

$$h_{Z,5}(x:y:z) = \begin{cases} +\frac{3}{4}\log_p(5) & \text{if } z \equiv 0 \bmod 5, \\ -\frac{3}{4}\log_p(5) & \text{if } x/z \equiv 2 \bmod 5, \\ 0 & \text{otherwise.} \end{cases}$$

Application: quadratic Chabauty

$$C : y^2 = x^6 + \frac{18}{5}x^4 + \frac{6}{5}x^3 + \frac{9}{5}x^2 + \frac{6}{5}x + \frac{1}{5}$$

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$$h_{Z,5}(x:y:z) = \begin{cases} +\frac{3}{4}\log_p(5) & \text{if } z \equiv 0 \bmod 5, \\ -\frac{3}{4}\log_p(5) & \text{if } x/z \equiv 2 \bmod 5, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem [Betts—D—Hashimoto—Spelier, '25]. The rational points on C are the 10 points:

$\{(-1/3 : -1/27 : 1), (-1/3 : 1/27 : 1), (-1/5 : -21/125 : 1), (-1/5 : 21/125 : 1),$
$(1 : 3 : 1), (1 : -3 : 1), (1 : 1 : 0), (1 : -1 : 0), (-1/2 : -3/8 : 1), (-1/2 : 3/8 : 1)\}$

Our method

Our implementation handles $\ell \neq p, 2$ for any hyperelliptic curve that:

1. is given by an affine model $y^2 = f(x)$, where $f(x)$ is a separable polynomial and has even degree > 3;
2. its Jacobian has a nontrivial endomorphism.

Our method

Our implementation handles $\ell \neq p, 2$ for any hyperelliptic curve that:

1. is given by an affine model $y^2 = f(x)$, where $f(x)$ is a separable polynomial and has even degree > 3;
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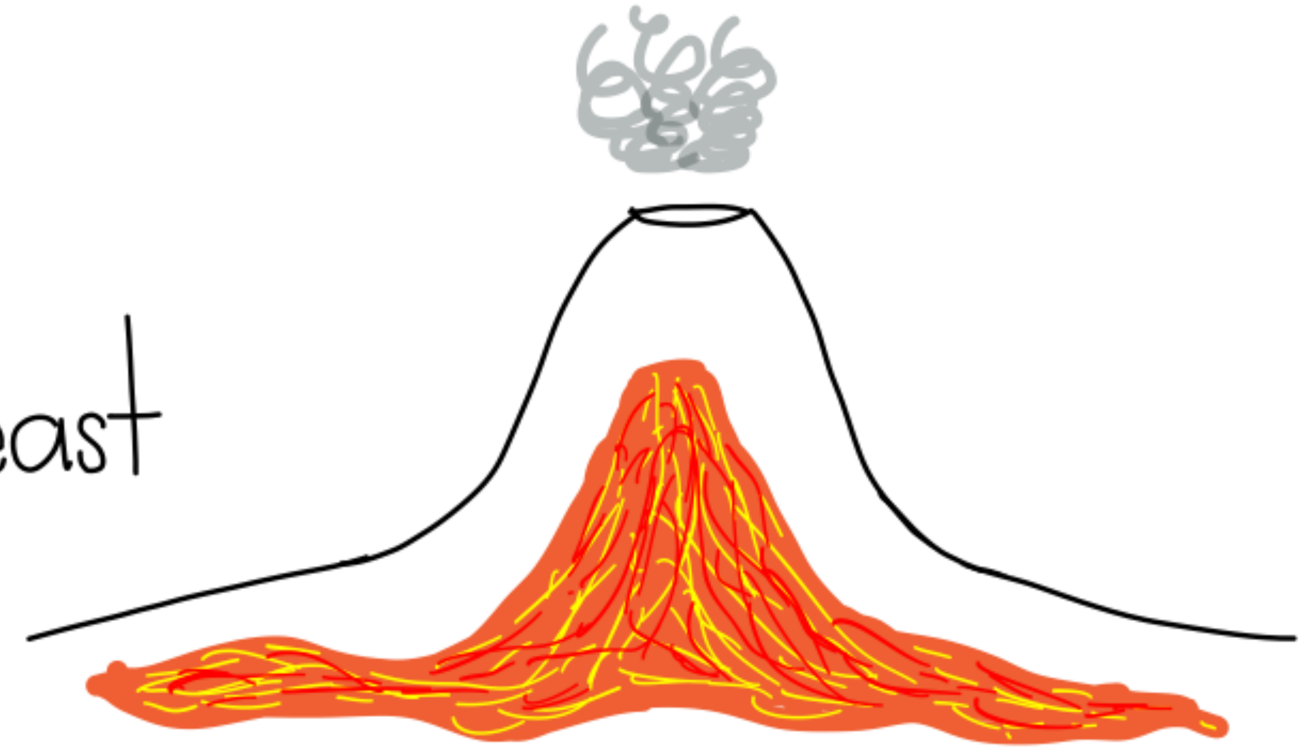
available at your neareast



Consequences of our method

- If $C/\mathbb{Q}_\ell$ is a genus 2 curve with the cluster picture

  and with leading coefficient a unit in $\mathbb{Z}_\ell$, then the local height of any $\mathbb{Q}_\ell$-point on C is 0.

- The Shimura curve quotient $X_0(93,1)/\langle \omega_{93} \rangle$ has trivial local height at 31.

- The Atkin-Lehner quotients $X_0(330)^*$, $X_0(255)^*$, and $X_0(147)^*$ have trivial local heights away from p.

Generalization: plane quartic

$$xy^3 + 2xy^2 + (x - x^3 - p)y - p = 0$$

Recall that we need to understand:

1. A semistable covering of C.

   3 to 1 cover of $\mathbb{P}^1$. ✓

2. The induced action of Z_* on the homology $H_1(\Gamma, \mathbb{Q})$ of the dual graph Γ of the geometric special fibre.

   Coleman-Iovita.

3. The traces $\mathrm{tr}_v(Z)$ attached to vertices of Γ.

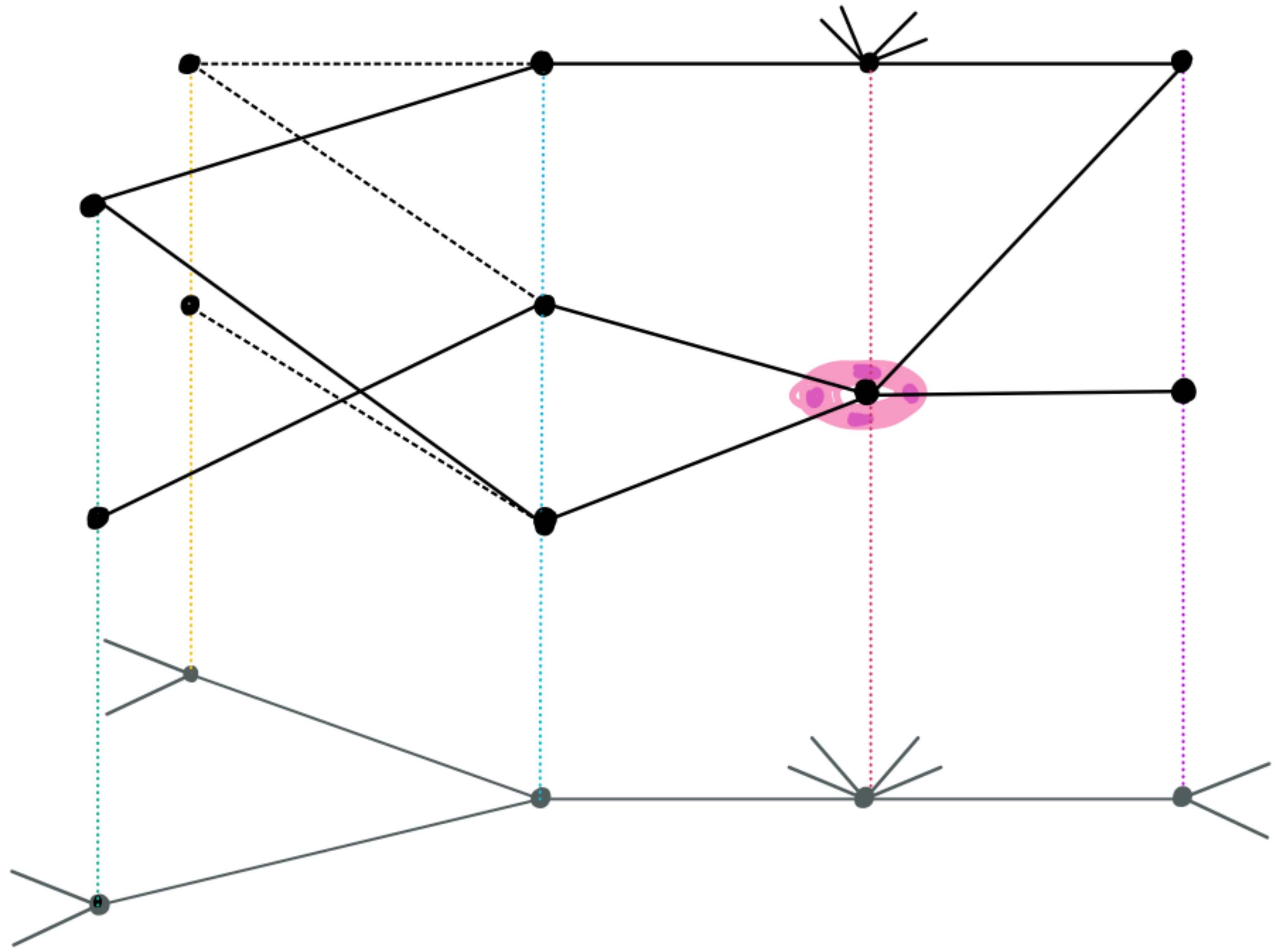
   Local/global symbols and push/pull formula.

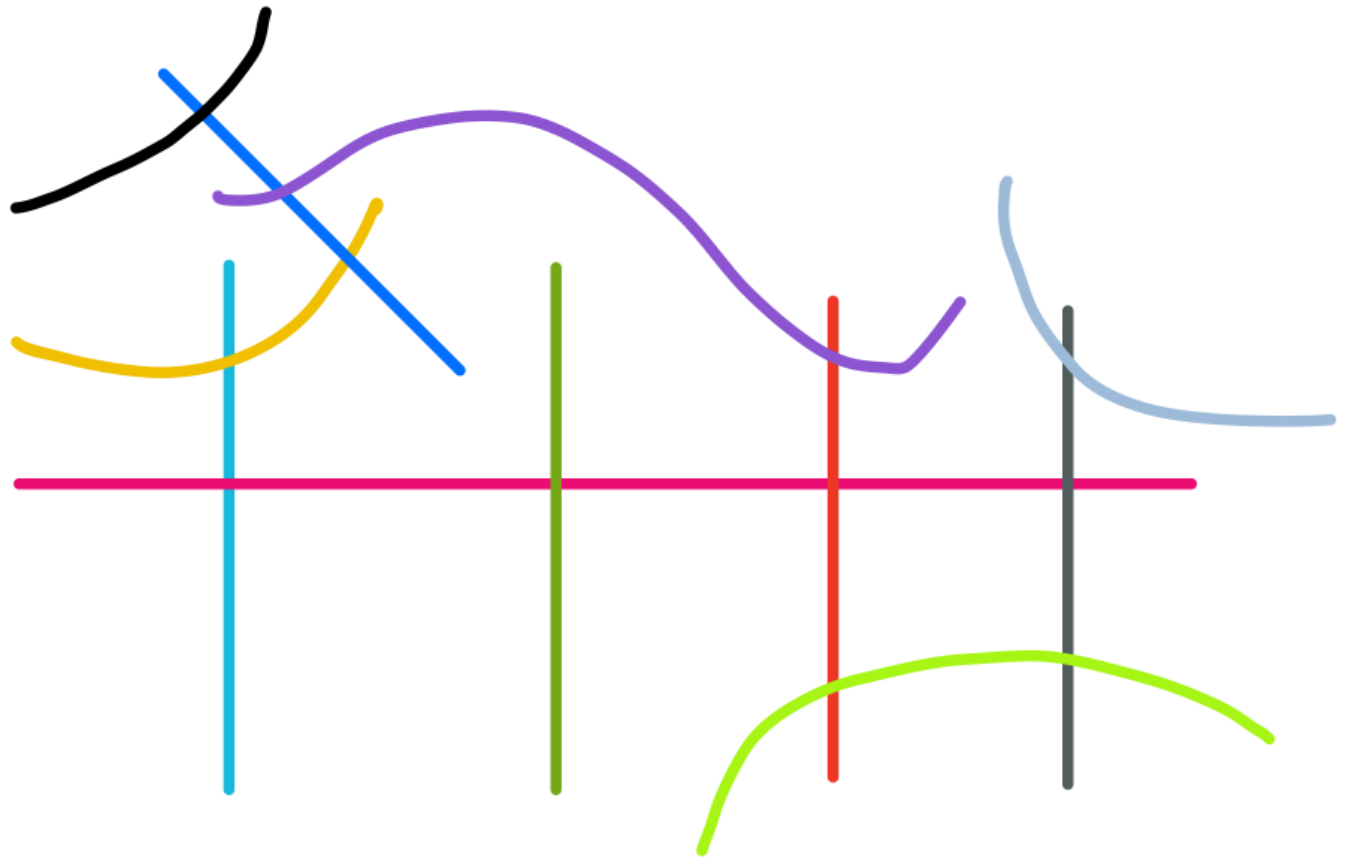
Generalization: plane quartic

$$xy^3 + 2xy^2 + (x - x^3 - p)y - p = 0$$

1. A semistable covering of C. 3 to 1 cover of $\mathbb{P}^1$. ✓

We follow ideas of Helminck '21 and Ossen '24.





Thank you!

To compute $h_{Z,\ell}$ for hyperelliptic curves, we find

1. A semistable covering of C. — Cluster pictures.
2. The induced action of Z_* on the homology $H_1(\Gamma, \mathbb{Q})$ of the dual graph Γ of the geometric special fibre. — Coleman-Iovita.
3. The traces $\mathrm{tr}_v(Z)$ attached to vertices of Γ. — Local/global symbols and push/pull formula.