Hilbert Modular Surfaces: An Introduction

① Motivation.

② Set up.

③ The Hilbert modular group, congruence subgroups.

④ Hilbert modular varieties.

⑤ Hilbert modular forms.

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Break

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Discussion.

⓪ Motivation: Modular Curves / Forms

Modular curves.

- Riemann surfaces obtained from quotients of $\hat{h}$ by the action of congruence subgroups.
- Parametrize isomorphism classes of elliptic curves.

Modular group: $SL_2(\mathbb{Z})$

Modular forms.

- Complex analytic functions on $\hat{h}$ satisfying a particular equation for the action of a congruence subgroup on $\hat{h}$.

$$f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} z\right) = (cz+d)^k f(z)$$

$$\Omega^k(X(\Gamma)) \cong \left\{ \begin{array}{l} \text{Meromorphic mod forms} \\ \text{of weight } \tfrac{k}{2} \text{ wrt } \Gamma \end{array} \right\}$$

Idea: Generalize this for powers $\hat{h}$.

① Set Up

- F is a totally real number field, $[F:\mathbb{Q}]=n$. ⋆ we allow $n=1$
- v represents a real place of F.
- $a\in F^\times$ is totally positive if $v(a)>0$ $\forall v$. same as $a\in \ker \mathrm{sgn}$.

  we write $a\in F^\times_{>0}$ $(a>>0)$
- $R:=\mathbb{Z}_F$ and $h:=\#\mathrm{Cl}\,R$.
- Cl^+R is the Narrow class group, $h^+:=\#\mathrm{Cl}^+R$

  $1\longrightarrow \{\pm1\}^n/\mathrm{sgn}(R^\times)\longrightarrow \mathrm{Cl}^+(R)\longrightarrow \mathrm{Cl}(R)\longrightarrow 1$

  Frac Ideals / Principal ideals generated by totally + elements
- $\mathfrak{h}$ is the upper half-plane, $\mathcal{H}:=(\mathfrak{h})^n$

  with the hyperbolic metric

② The Hilbert Modular Group

Action on $\mathcal{H} = (\mathfrak{h})^n$:

$$PGL_2^+(F) := \{\gamma \in GL_2(F) \mid \det\gamma \in F^\times_{>0}\} / \left\{\begin{pmatrix} a & 0 \\ 0 & a\end{pmatrix} : a \in F\right\} \quad GL_2^+(F) \circlearrowright \mathcal{H}$$

$$z \in \mathcal{H} \qquad z = (z_v)_v \longmapsto \gamma z := \left(\frac{a_v z_v + b_v}{c_v z_v + d_v}\right)_v \qquad a_v := v(a)$$

$$\Gamma_F := PSL_2(R) \subseteq PGL_2^+(F)$$

Cusps.

We have an embedding: $\mathbb{P}^1(F) \longrightarrow \mathbb{P}^1(\mathbb{R})^n \subset \mathbb{P}^1(\mathbb{C})^n$, with $PGL_2^+(F) \circlearrowright \mathbb{P}^1(F)$ and $PGL_2^+(\mathbb{R}) \circlearrowright \mathbb{P}^1(\mathbb{R})^n$.

The orbits of $\mathbb{P}^1(F)$ under Γ_F are called cusps of Γ_F.

$[\alpha : \beta] \in \mathbb{P}^1(F)$ with $\alpha, \beta \in R$ $\qquad [\alpha : \beta] \rightsquigarrow (\alpha, \beta) \subseteq Cl(F)$

Bijection!!

Fact: The correspondence $[\alpha:\beta] \leftrightsquigarrow (\alpha,\beta) \subseteq Cl(F)$ is a bijection

$\begin{pmatrix} \alpha & \alpha^* \\ \beta & \beta^* \end{pmatrix}$ transforms $[1:0]$ to $[\alpha:\beta]$ $\quad \alpha\beta^* - \alpha^*\beta = 1$

$\mathfrak{b} \in Frac(R)$

$$SL_2(R \oplus \mathfrak{b}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(F) : a, d \in R,\ b \in \mathfrak{b}^{-1},\ c \in \mathfrak{b} \right\}$$

Congruence Subgroups.

$N \subseteq R$ nonzero ideal, $\mathfrak{b} \subset F$ nonzero fractional R-ideal

$$\Gamma_0(N)_{\mathfrak{b}} := \begin{pmatrix} R & \mathfrak{b}^{-1} \\ N\mathfrak{b} & R \end{pmatrix} \cap GL_2^+(F) \cap \det^{-1}(R^\times)$$

In particular $\Gamma_0(1)_{\mathfrak{b}} = GL^+(R \oplus \mathfrak{b}) = Aut_R^+(R \oplus \mathfrak{b})$

Oriented R-mod aut. of $R \oplus \mathfrak{b}$

$$\Gamma_1(N)_{\mathfrak{b}} := \begin{pmatrix} 1+N & \mathfrak{b}^{-1} \\ N\mathfrak{b} & R \end{pmatrix} \cap GL_2^+(F) \cap \det^{-1}(R^\times)$$

$$\Gamma(N)_{\mathfrak{b}} := \begin{pmatrix} 1+N & N\mathfrak{b}^{-1} \\ N\mathfrak{b} & 1+N \end{pmatrix} \cap GL_2^+(F) \cap \det^{-1}(R^\times)$$

A congruence subgroup $\Gamma \leq GL_2^+(F)$ is a subgroup conjugate to a group that contains $\Gamma(N)_{\mathfrak{b}}$ for some N and $\mathfrak{b}$.

Isomorphisms of congruence subgroups.

$\alpha \in F_{>0}^{\times}$.

$$\begin{pmatrix} \alpha^{-1} & 0 \\ 0 & 1 \end{pmatrix} \Gamma_0(N)_{\mathfrak{b}} \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} = \Gamma_0(N)_{\alpha\mathfrak{b}}.$$

We only need to understand groups $\Gamma_0(N)_{\mathfrak{b}}$ for choices of reps of $Cl^+(R)$.

⑤ Hilbert Modular Varieties

$\Gamma_{\mathfrak{b}} < GL_2^+(F)$ congruence subgroup

$$Y(\Gamma_{\mathfrak{b}})(\mathbb{C}) := \Gamma_{\mathfrak{b}} \backslash \mathcal{H}$$ complex orbifold of dim n.

Can be seen as the moduli space of polarized abelian surfaces.

★ Moduli interpretations for non-parallel weight?

$$\overline{Y}(\Gamma_{\mathfrak{b}})(\mathbb{C}) := \Gamma_{\mathfrak{b}} \backslash \mathcal{H}^*$$ is compact

One can construct a minimal desingularization

$$\pi : X(\Gamma_{\mathfrak{b}}) \to \overline{Y}(\Gamma_{\mathfrak{b}})$$

using continued fractions. (For quadratic fields).

We also define: $X_0(N) := \bigsqcup_{\mathfrak{b} \in Cl^+(F)} X_0(N)_{\mathfrak{b}}$, For $\Gamma_0(N)_{\mathfrak{b}}$ $\quad X_1(N) := \bigsqcup_{\mathfrak{b} \in Cl^+(F)} X_1(N)_{\mathfrak{b}}$ $\quad \Gamma_1(N)_{\mathfrak{b}}$.

Theorem (Baily-Borel) The complex analytic space $\overline{\Gamma \backslash \mathcal{H}}$ is the normal complex analytic space of $\mathrm{Proj}(M(\Gamma))$.

↑ Hilbert Modular forms of Γ

Hilbert Modular Surfaces: $n=2$ Quadratic extensions of $\mathbb{Q}$

⑥ Hilbert Modular Forms

Let $k = (k_v)_v \in \mathbb{Z}_{\geq 0}^n$ with all k_i of the same parity. can be relaxed: Non-paritious HMF's

A holomorphic function $f : \mathcal{H} \to \mathbb{C}$ is a Hilbert modular form of weight k on Γ if for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$,

$$f(\gamma z) = \left(\prod_v \frac{(c_v z_v + d_v)^{k_v}}{\det(\gamma_v)^{k_v/2}} \right) f(z)$$

$(z_v)_v$ → z

$\underbrace{\prod_v \frac{(c_v z_v + d_v)^{k_v}}{\det(\gamma_v)^{k_v/2}}}$ $!!$ $\frac{(cz+d)^k}{\det(\gamma)^{k/2}}$

together with f being holomorphic at cusps if $n=1$.

For $\alpha \in GL_2^+(\mathbb{R})^n$, $(f|_k \alpha)(z) := \frac{\det(\alpha)^{k/2}}{(cz+d)^k} f(\alpha z)$.

$f|_k \gamma = f \quad \forall \gamma \in \Gamma$

The weight is parallel when all the k_v's are equal.

HMFs.

$M_k(\Gamma_{\mathfrak{b}})$ is the $\mathbb{C}$-vector space of Hilbert modular forms of weight k for $\Gamma_{\mathfrak{b}}$. Finite dim.

For parallel weights: $M(\Gamma_{\mathfrak{b}}) := \bigoplus_{k \in \mathbb{Z}_{\geq 0}} M_k(\Gamma_{\mathfrak{b}})$ is a graded ring.

For not parallel weights: $M(\Gamma_{\mathfrak{b}})$ might not even be finitely generated

★ Exercise: Try non parallel weight $(k, 2k)$ and see if it is finitelly generated.

Also, $M_k(\Gamma) := \bigoplus_{\mathfrak{b} \in Cl^+(R)} M_k(\Gamma_{\mathfrak{b}})$

Examples. Eisenstein series of weight $2r$

Theta series (Come from quadratic forms)

Cusp forms. HMF's f such that $f(z) \to 0$ as $z \to c$ for all cusps c of $\Gamma_{\mathfrak{b}}$. $S_k(\Gamma_{\mathfrak{b}}) \subseteq M_k(\Gamma_{\mathfrak{b}})$

Fourier Expansions.

Let $\mathfrak{a} \subseteq F$ be a fractional ideal, the dual of $\mathfrak{a}$ under the trace pairing is:

$$\mathfrak{a}^{\#} := \{x \in F \mid Tr(x\mathfrak{a}) \subseteq \mathbb{Z}\}$$

$R^{\#}$ is the codifferent. $\mathfrak{d}_F := (R^{\#})^{-1}$ is the different

$R^{\#} \supseteq R$

Relation: $\mathfrak{a}^{\#} = \mathfrak{a}^{-1} R^{\#} = \mathfrak{a}^{-1} \mathfrak{d}_F^{-1}$.

Note: $Nm(\mathfrak{d}_F) = d_F \mathbb{Z}$, $d_F = disc(F/\mathbb{Q})$

Let $\Gamma_{\mathfrak{b}} = \Gamma_1(N)_{\mathfrak{b}}$. The stabilizer of $[1:0] \in \mathbb{P}^1(F)$ under $\Gamma_{\mathfrak{b}}$ is:

$$\begin{pmatrix} R & \mathfrak{b}^{-1} \\ 0 & R \end{pmatrix} \cap GL_2^+(F) \cap \det^{-1}(R^\times)$$

f HMF, then f is invariant under $z \to z + b$ for $b \in \mathfrak{b}^{-1}$, so f has a Fourier expansion over the dual of $\mathfrak{b}^{-1}$.

$(\mathfrak{d}^{-1})^{\#} = \mathfrak{d}R^{\#} = \mathfrak{d}\mathfrak{d}_F^{-1}$

The Fourier expansion for $f \in M(\Gamma_{\mathfrak{d}})$ can be written as:

$$f(z) = a_0 + \sum_{\nu \in (\mathfrak{d}\mathfrak{d}_F^{-1})_{>0}} a_\nu \exp(2\pi i(\nu_1 z_1 + \cdots + \nu_n z_n))$$

a_0: $=0$ cusps; $a_\nu \in \mathbb{C}$; $\exp(\ldots) = q^{Tr(\nu z)}$

More relations: $\varepsilon \in R^{\times}_{>0}$ totally positive unit. $\begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix} \in \Gamma_{\mathfrak{d}}$

$$f(\varepsilon^{-1} z) = f((\varepsilon_\nu^{-1} z_\nu)_\nu) = \left(\prod_\nu \frac{\varepsilon_\nu^{k_\nu}}{\varepsilon_\nu^{k_\nu/2}}\right) f(z) = \prod_\nu (\varepsilon^{k_\nu/2})_\nu f(z)$$

($c_\nu z_\nu + d_\nu$; det $\begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix}$)

$$\varepsilon^{k/2} := \prod_\nu \varepsilon^{k_\nu/2}$$

$$a_0 + \sum_{\nu \in (\mathfrak{d}\mathfrak{d}_F^{-1})_{>0}} a_\nu q^{Tr(\nu\varepsilon^{-1} z)} = a_0 + \sum_{\nu \in (\mathfrak{d}\mathfrak{d}_F^{-1})_{>0}} a_{\nu\varepsilon} q^{Tr(\nu z)} = \varepsilon^{k/2} a_0 + \sum_{\nu \in (\mathfrak{d}\mathfrak{d}_F^{-1})_{>0}} \varepsilon^{k/2} a_\nu q^{Tr(\nu z)}$$

$a_0 = \varepsilon^{k/2} a_0$ $\qquad$ $a_{\nu\varepsilon} = \varepsilon^{k/2} a_\nu$

If $k \in 2\mathbb{Z}_{\geq 0}^n$ is parallel, then $\varepsilon^{k/2} = 1$ because $Nm(\varepsilon) = 1 \in \mathbb{Z}_{>0}^\times$

k not parallel $\Rightarrow \varepsilon \neq 1 \Rightarrow \varepsilon^{k/2} \neq 1 \Rightarrow a_0 = 0$

Every HMF in non-parallel weight is a cusp form.

What was special about $\varepsilon^{k/2}$?

Let $w: \mathbb{R}_{>0}^{\times} \longrightarrow \mathbb{C}^{\times}$ be a group homomorphism.

We say that w is the unit character of a form and ask that $a_{v_\varepsilon} = w(\varepsilon) a_v \quad \forall v$

$\varepsilon \in \mathbb{R}_{>0}^{\times}$

$\begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix}$ normalizes Γ_{Δ} (← $\Gamma_1(N)_{\Delta}$) and belongs to $P\Gamma_{\Delta}$ iff $\varepsilon \in \mathbb{R}^{\times 2}$

If $\varepsilon = u^2$ $\quad \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix} = u \begin{pmatrix} u^{-1} & 0 \\ 0 & u \end{pmatrix}$

$w_0 : \mathbb{R}_{>0}^{\times} / \mathbb{R}^{\times 2} \longrightarrow \{\pm 1\}$

$w(\varepsilon) := \varepsilon^{k/2} w_0(\varepsilon)$ decomposes the space of SL_2 using this unit character.

★ Rewrite this better in the Overleaf.

Characters.

φ_0 character of $(R/N)^\times \simeq \Gamma_0(N)_{\mathfrak{b}} / \Gamma_1(N)_{\mathfrak{b}}$ such that $\forall \varepsilon \in R^\times$,
$\varphi_0(\varepsilon) = \mathrm{sgn}(\varepsilon)^k$.

A Hilbert modular form on $\Gamma_0(N)_{\mathfrak{b}}$ of weight k and character φ_0 is a holomorphic function $f: \mathcal{H} \to \mathbb{C}$ s.t.

$$f|_k \gamma = \varphi_0(\gamma) f \quad \forall \gamma \in \Gamma_0(N)_{\mathfrak{b}}$$

Before $f|_k \gamma = f$