Origami and Mathematics

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December 12, 2019

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Robert J. Lang

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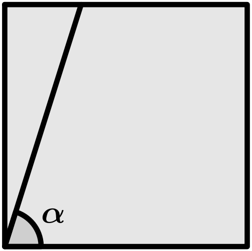

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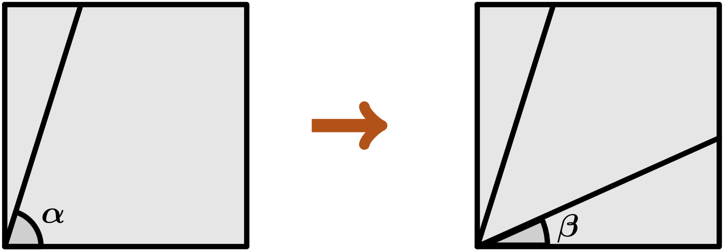

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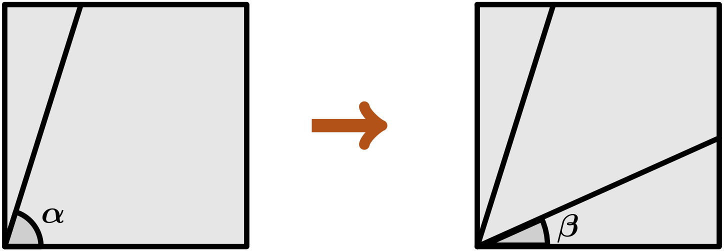
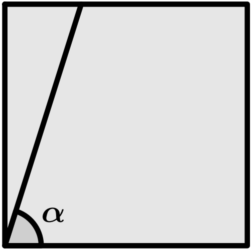


This shows how origami is more powerful than straightedge and compass.

How to do it?



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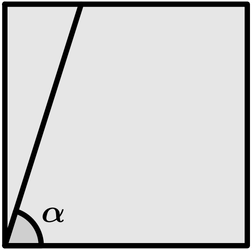
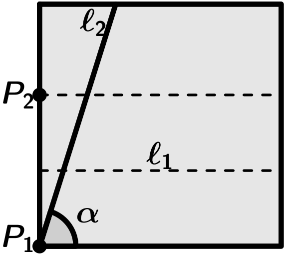
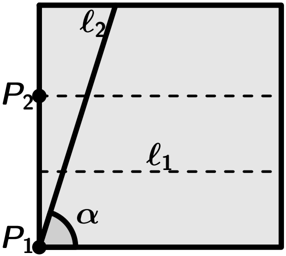
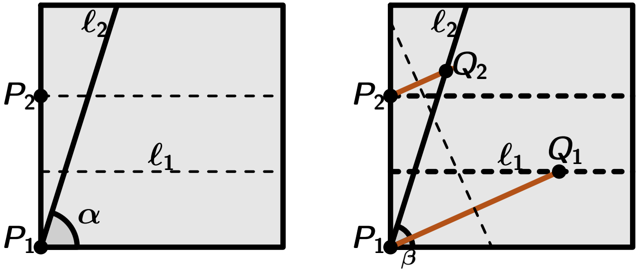






ℓ2
P2
ℓ1
α
P1


ℓ_2
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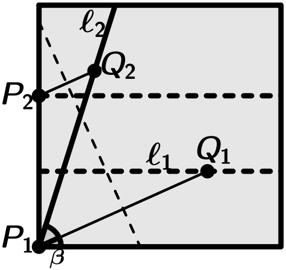
α
P_1
ℓ_2
Q_2
P_2
ℓ_1
Q_1
P_1
β


ℓ_2
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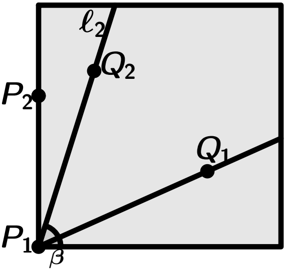
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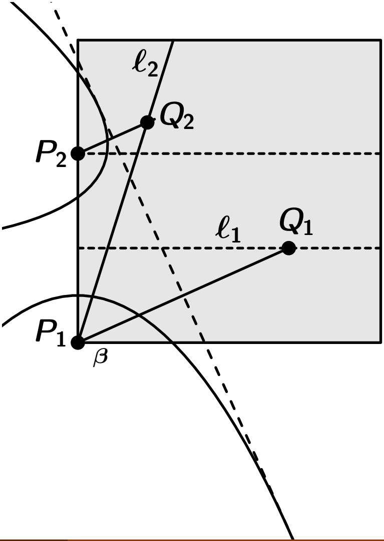
What is happening?

We are finding simultaneous tangents to parabolas.



An algebra application: solving $x^3 + ax + b$

We will find the solutions for $\boldsymbol{x^3 + ax + b = 0}$ where $a,\ b \in \mathbb{R}$ and $b \neq 0$ by finding a simultaneous tangent to:

$$\left(y - \frac{1}{2}a\right)^2 = 2bx \qquad \text{and} \qquad y = \frac{1}{2}x^2$$

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Hence:

$$\begin{aligned} x_1 &= \frac{b}{2m_1^2} \\ y_1 &= \frac{b}{m_1} + \frac{a}{2} \end{aligned} \qquad \text{and} \qquad \begin{aligned} x_2 &= m_2 \\ y_2 &= \frac{m_2^2}{2} \end{aligned}$$

$$x_1 = \frac{b}{2m^2} \qquad \text{and} \qquad x_2 = m$$

$$y_1 = \frac{b}{m} + \frac{a}{2} \qquad\qquad y_2 = \frac{m^2}{2}$$

So the slope of the line between these points is:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{m^4 - 2bm - am^2}{2m^3 - b}$$

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$$m(m^3 + am + b) = 0$$

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Solutions for cubic polynomials

Real roots of $\boldsymbol{x^3 + ax + b}$ are the slope of a simultaneous tangent to:

$$\left(y - \frac{1}{2}a\right)^2 = 2bx \qquad \text{and} \qquad y = \frac{1}{2}x^2$$

Example: $a = 2$ and $b = 1$

$$(y-1)^2 = 2x \qquad \text{and} \qquad y = \frac{1}{2}x^2$$

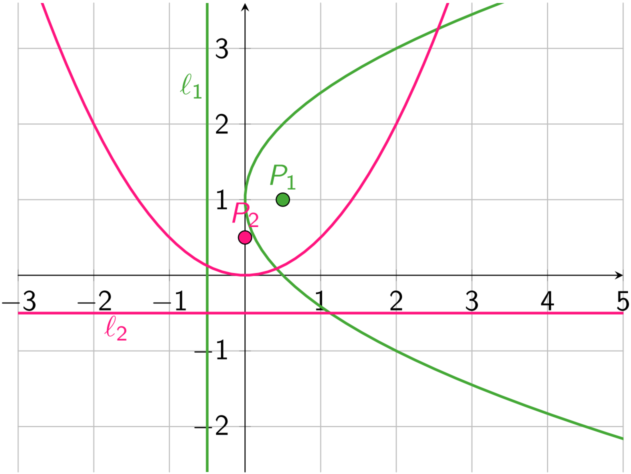
Directrix

$$x = -\frac{1}{2} \qquad\qquad y = -\frac{1}{2}$$

Focus

$$\left(\frac{1}{2}, 1\right) \qquad\qquad \left(0, \frac{1}{2}\right)$$

$$(y-1)^2 = 2x \qquad y = \frac{1}{2}x^2$$



The method

$P_1 = (0.5, 1)$, $\ell_1 : x = -0.5$

$P_2 = (0, 0.5)$, $\ell_2 : y = -0.5$

1. **Construct the x and y axis.**
2. Identify P_1, P_2, ℓ_1 and ℓ_2 in the paper.
3. Make a fold such that P_1 touches ℓ_1 and P_1 touches ℓ_1 **at the same time.** The slope m of the resulting line is the solution.
4. Find the point $(m, 0)$

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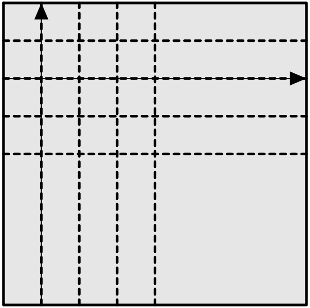

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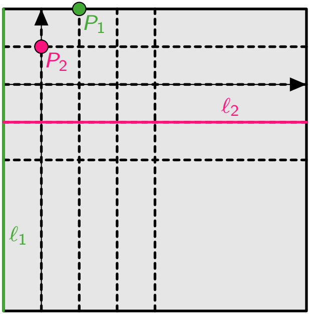

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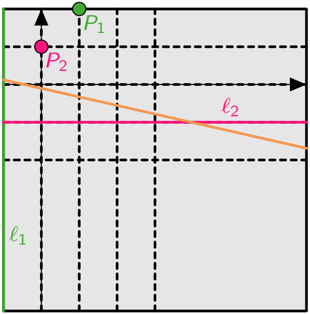

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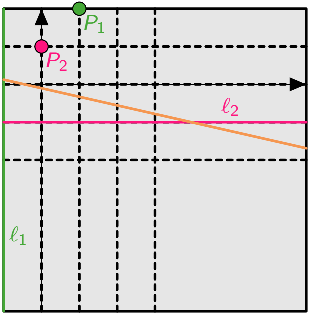
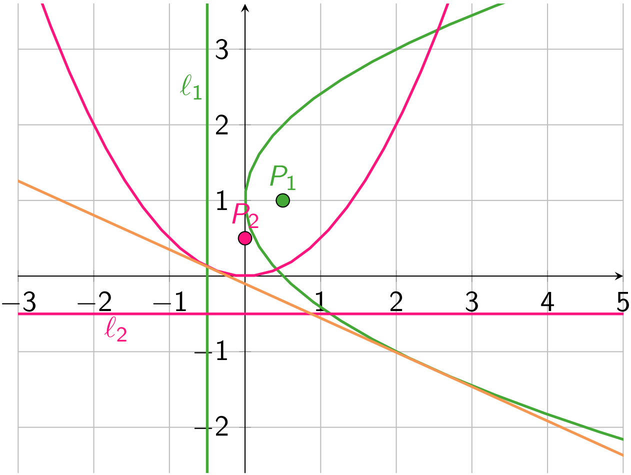


$$(y-1)^2 = 2x \qquad y = \frac{1}{2}x^2$$



$$y \approx \mathbf{-0.4534}x - 0.102786$$

Remark

The real solution of $x^3 + 2x + 1$ is **not** rational:

$$x = \frac{\sqrt[3]{\frac{1}{2}\left(\sqrt{177} - 9\right)}}{3^{2/3}} - 2\sqrt[3]{\frac{2}{3\left(\sqrt{177} - 9\right)}}$$

Huzita Axioms

1. Given two points P_1 and P_2 there is a unique fold passing through both of them.
2. Given two points P_1 and P_2 there is a unique fold placing P_1 onto P_2.
3. Given two lines L_1 and L_2, there is a fold placing L_1 onto L_2.
4. Given a point P and a line L, there is a unique fold perpendicular to L passing through P.
5. Given two points P_1 and P_2 and a line L, there is a fold placing P_1 onto L and passing through P_2.
6. Given two points P_1 and P_2 and two lines L_1 and L_2, there is a fold placing P_1 onto L_1 and P_2 onto L_2.
7. Given a point P and two lines L_1 and L_2, there is a fold placing P onto L_1 and perpendicular to L_2.

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Let $\mathscr{O}$ be the set of numbers that are constructible using **origami**.

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$$\mathscr{A} \subsetneq \mathscr{O}$$

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$$\alpha \in \mathscr{O} \iff \alpha \text{ is constructible by marked ruler}$$

$$\iff \alpha \text{ is constructible by intersecting conics}$$

$$\iff \alpha \text{ lies on a 2-3 tower } \mathbb{Q} = F_0 \subseteq F_1 \subset \cdots \subset F_n$$

$$\iff \alpha \text{ is algebraic over } \mathbb{Q} \text{ with minimal polynomial of degree } 2^k 3^l$$

Thank you!