Origami and Math

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Fun fact

What are ”nice” properties of paper?

- Cannot be stretched or compressed.
- Cannot be sheared.
- It can be easily folded!

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Robert J. Lang

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- At each step of the folding, each parallelogram is completely **flat**. This means that we can use rigid materials.
- Folded material can be unpacked in **one motion** by pulling on its opposite ends, and likewise folded by pushing the two ends together.

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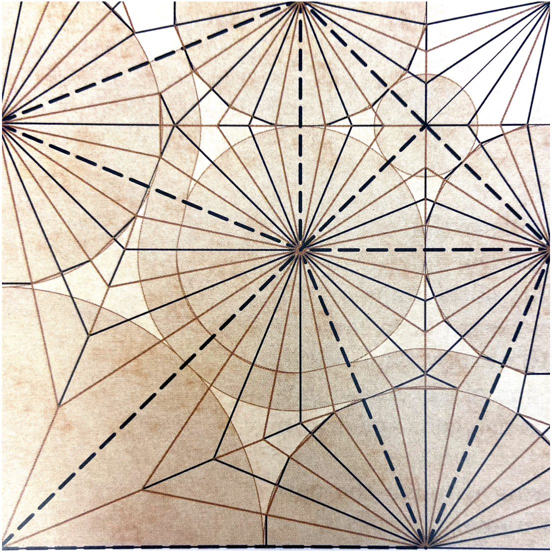
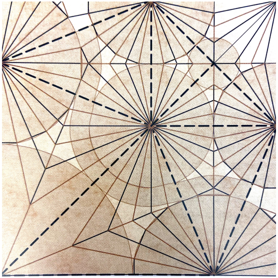
Circle/river method or tree method (1994): Gives a systematic method for folding any structure that topologically resembles a weighted tree.

Circle packing



Circle placing

Given a set of n circles, place the circle centers on the paper, such that the overall circle layout is non-overlapping.



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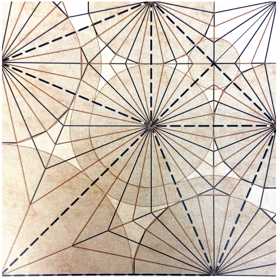

The problem of Circle Placing is **NP-Hard**.

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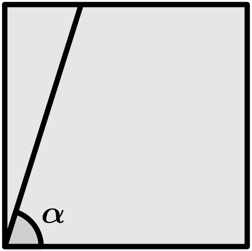

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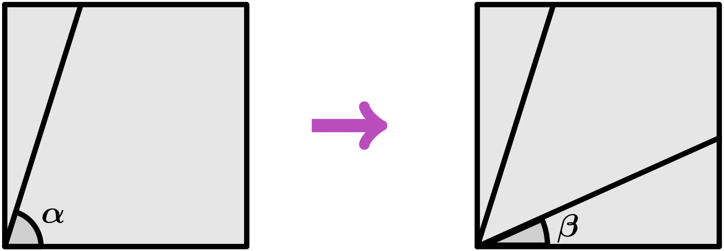

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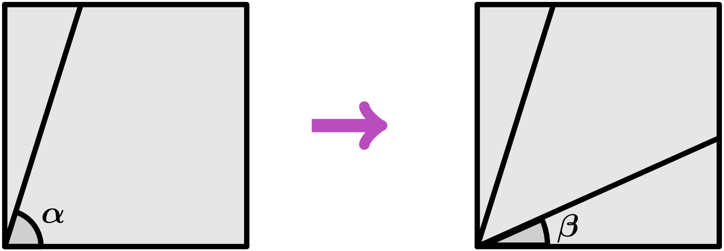
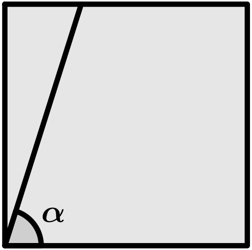


This shows how origami is more powerful than straightedge and compass.

How to do it?



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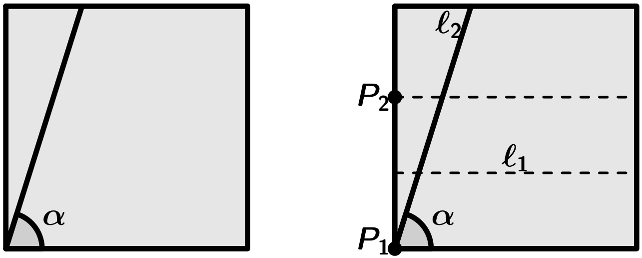
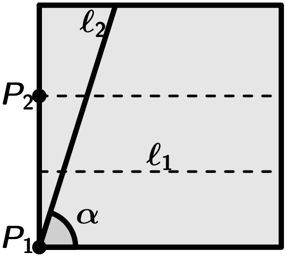
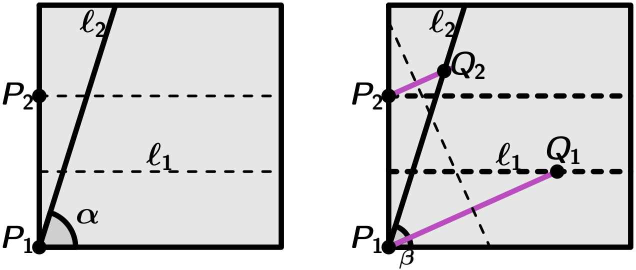



ℓ_2
P_2
ℓ_1
α
P_1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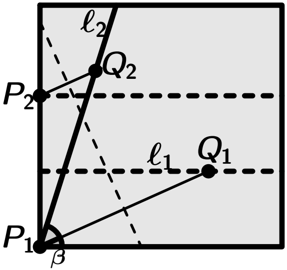

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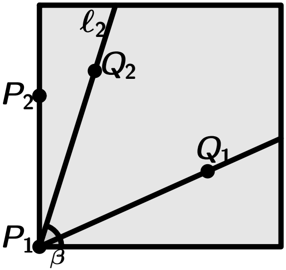

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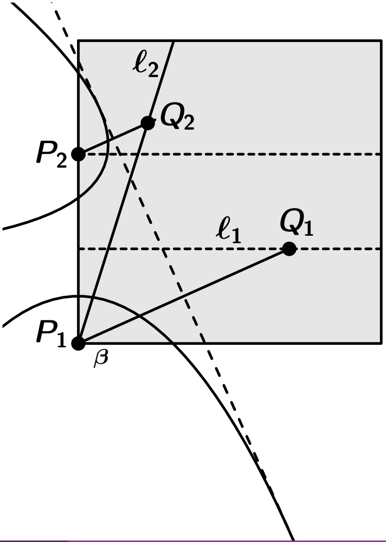
What is happening?

We are finding simultaneous tangents to parabolas.



An algebra application: solving $x^3 + ax + b$

We will find the solutions for $\boldsymbol{x^3 + ax + b = 0}$ where $a,\ b \in \mathbb{R}$ and $b \neq 0$ by finding a simultaneous tangent to:

$$\left(y - \frac{1}{2}a\right)^2 = 2bx \qquad \text{and} \qquad y = \frac{1}{2}x^2$$

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Hence:

$$\begin{aligned} x_1 &= \frac{b}{2m_1^2} \\ y_1 &= \frac{b}{m_1} + \frac{a}{2} \end{aligned} \qquad \text{and} \qquad \begin{aligned} x_2 &= m_2 \\ y_2 &= \frac{m_2^2}{2} \end{aligned}$$

$$x_1 = \frac{b}{2m^2} \qquad \qquad x_2 = m$$

and

$$y_1 = \frac{b}{m} + \frac{a}{2} \qquad \qquad y_2 = \frac{m^2}{2}$$

So the slope of the line between these points is:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{m^4 - 2bm - am^2}{2m^3 - b}$$

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$$m(m^3 + am + b) = 0$$

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Solutions for cubic polynomials

Real roots of $\boldsymbol{x^3 + ax + b}$ are the slope of a simultaneous tangent to:

$$\left(y - \frac{1}{2}a\right)^2 = 2bx \qquad \text{and} \qquad y = \frac{1}{2}x^2$$

Example: $a = 2$ and $b = 1$

$$(y-1)^2 = 2x \qquad \text{and} \qquad y = \frac{1}{2}x^2$$

Directrix

$$x = -\frac{1}{2} \qquad\qquad y = -\frac{1}{2}$$

Focus

$$\left(\frac{1}{2}, 1\right) \qquad\qquad \left(0, \frac{1}{2}\right)$$

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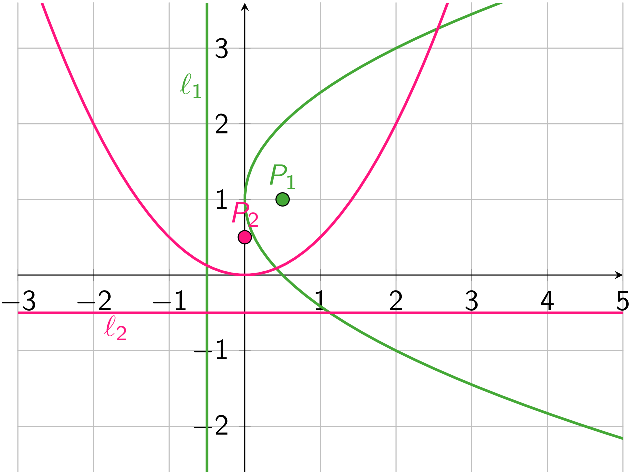

The method

$\boldsymbol{P_1} = (0.5, 1)$, $\boldsymbol{\ell_1} : x = -0.5$ $\boldsymbol{P_2} = (0, 0.5)$, $\boldsymbol{\ell_2} : y = -0.5$

1. **Construct the *x* and *y* axis.**
2. Identify P_1, P_2, ℓ_1 and ℓ_2 in the paper.
3. Make a fold such that P_1 touches ℓ_1 and P_1 touches ℓ_1 **at the same time.** The slope m of the resulting line is the solution.
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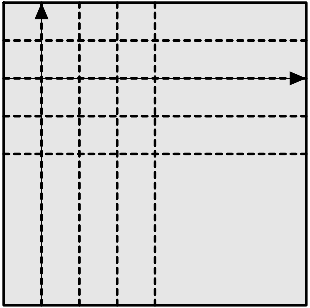

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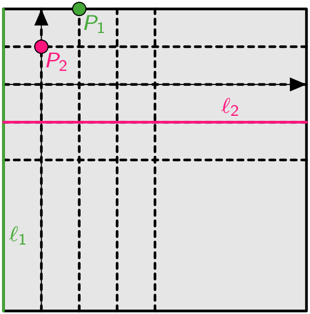

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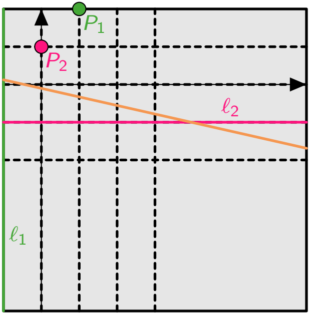

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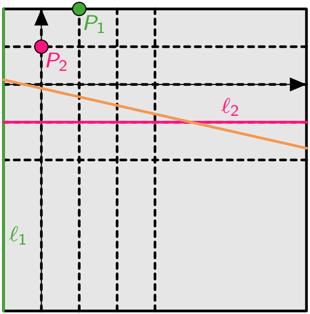

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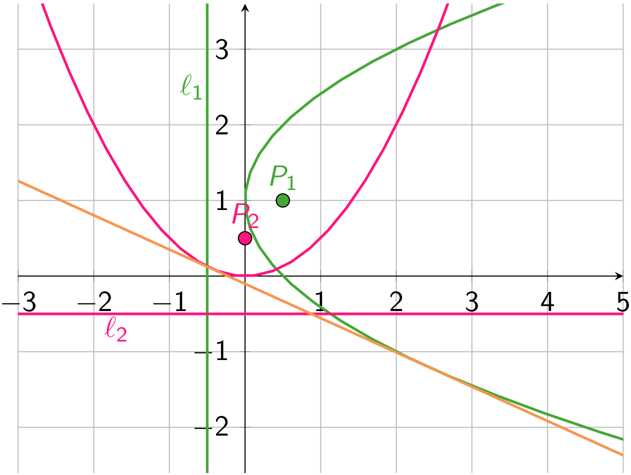

$$y \approx \mathbf{-0.4534}x - 0.102786$$

Remark

The real solution of $x^3 + 2x + 1$ is **not** rational:

$$x = \frac{\sqrt[3]{\frac{1}{2}\left(\sqrt{177} - 9\right)}}{3^{2/3}} - 2\sqrt[3]{\frac{2}{3\left(\sqrt{177} - 9\right)}}$$

Huzita Axioms

1. Given two points P_1 and P_2 there is a unique fold passing through both of them.
2. Given two points P_1 and P_2 there is a unique fold placing P_1 onto P_2.
3. Given two lines L_1 and L_2, there is a fold placing L_1 onto L_2.
4. Given a point P and a line L, there is a unique fold perpendicular to L passing through P.
5. Given two points P_1 and P_2 and a line L, there is a fold placing P_1 onto L and passing through P_2.
6. Given two points P_1 and P_2 and two lines L_1 and L_2, there is a fold placing P_1 onto L_1 and P_2 onto L_2.
7. Given a point P and two lines L_1 and L_2, there is a fold placing P onto L_1 and perpendicular to L_2.

Origami numbers

Let $\mathscr{O}$ be the set of numbers that are constructible using **origami**.

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$$\mathscr{A} \subsetneq \mathscr{O}$$

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$$\begin{aligned}
\alpha \in \mathscr{O} \quad &\iff \quad \alpha \text{ is constructible by marked ruler} \\
&\iff \quad \alpha \text{ is constructible by intersecting conics} \\
&\iff \quad \alpha \text{ lies on a 2-3 tower } \mathbb{Q} = F_0 \subseteq F_1 \subset \cdots \subset F_n \\
&\iff \quad \alpha \text{ is algebraic over } \mathbb{Q} \text{ with minimal polynomial of degree } 2^k 3^l
\end{aligned}$$