2.2 p-adic heights on Jacobians of curves

(Thanks to Grant and Sachi Hashimoto)

1. Why?

We can use p-adic heights $h(D_1, D_2)$ to study integral points.

Theorem. (Corollary 2.30) Suppose:

- $f(x) \in \mathbb{Z}[x]$, monic and separable, $\boxed{\deg f = 2g+1 > 3}$?
- $\mathcal{U} := \operatorname{Spec}(\mathbb{Z}[x,y]/(y^2 - f(x)))$,
- X normalization of the projective closure of the generic fiber of $\mathcal{U}$,
- $J := \operatorname{Jac}(X)$, $\operatorname{rk} J(\mathbb{Q}) = g$,
- p prime of good reduction such that
$$J(\mathbb{Q}) \otimes \mathbb{Q}_p \xrightarrow{\sim} H^0(X_{\mathbb{Q}_p}, \Omega^1)^*.$$

Then there are computable constants $\alpha_{ij} \in \mathbb{Q}_p$ s.t.

$$\rho(z) := \underbrace{\theta(z)}_{h_p(z-\infty,\, z-\infty)} - \sum_{0 \le i \le j \le g-1} \alpha_{ij} \int_\infty^z \omega_i \int_\infty^z \omega_j$$

vanishes on $\mathcal{U}(\mathbb{Z}[1/p])$.

Main idea of the proof: decompose $h(P-\infty, P-\infty)$ in two ways.

Algorithm. (2.31) **Input:** $X/\mathbb{Q}$ as in the previous theorem.
Output: The set of integral points on X.

2. Definition

Set up.

- $X/\mathbb{Q}$ nice curve of genus $g \geq 1$.
- p prime of good reduction.
- Fix the following:
  - a branch $\log_p : \mathbb{Q}_p^* \longrightarrow \mathbb{Q}_p$
  - an idèle class character $\chi : \mathbb{A}_{\mathbb{Q}}^*/\mathbb{Q}^* \longrightarrow \mathbb{Q}_p$

    (continuous homomorphism that decomposes as a sum of local characters).
  - A splitting s of the Hodge filtration on $H^1_{dR}(X/\mathbb{Q}_p)$ such that $\ker(s)$ is isotropic with respect to the cup product pairing.
  - A basis for $H^1_{dR}(X)$, $\{\omega_0, \ldots, \omega_{2g-1}\}$, with $\{\omega_0, \ldots, \omega_{g-1}\} \in H^0(X_{\mathbb{Q}_p}, \Omega^1)$.
  - A lift ϕ of Frobenious

Definition (Coleman-Gross). The cyclotomic p-adic height pairing is a symmetric bi-additive pairing

$$\begin{aligned} \mathrm{Div}^0(X) \times \mathrm{Div}^0(X) &\longrightarrow \mathbb{Q}_p \\ (\underbrace{D_1 , \quad D_2}_{\text{Disjoint support}}) &\longmapsto h(D_1, D_2) \end{aligned}$$

such that

(1) $$h(D_1, D_2) = \sum_{\substack{\text{finite} \\ \text{primes } v}} h_v(D_1, D_2) \quad + 0$$

$$= h_p(D_1, D_2) + \sum_{l \neq p} h_l(D_1, D_2)$$

$$= \underbrace{\int_{D_2} \omega_{D_1}}_{\text{Coleman integral}} \quad + \sum_{l \neq p} \underbrace{m_l}_{\in \mathbb{Q}, \text{ Intersection mult.}} \log_p l .$$

(2) For $\beta \in \mathbb{Q}(X)^*$, we have

$$h(D, \mathrm{div}(\beta)) = 0$$

Note: Part (2) implies that the induced pairing

$$h : J(\mathbb{Q}) \times J(\mathbb{Q}) \longrightarrow \mathbb{Q}_p$$

is a bilinear pairing.

3. $h_p(D_1, D_2) = \int_{D_2} w_{D_1}$

Construction of w_{D_1}

$T(\mathbb{Q}_p) := \left\{ \begin{array}{c} \text{Differentials with at most simple poles and} \\ \text{integer residues} \end{array} \right\}$

$$\begin{aligned} \mathrm{Res}: T(\mathbb{Q}_p) &\longrightarrow \mathrm{Div}^0(X) \\ w &\longmapsto \sum_P (\mathrm{Res}_P(w)) P. \end{aligned}$$

Induces

$$0 \longrightarrow H^0(X_{\mathbb{Q}_p}, \Omega^1) \longrightarrow T(\mathbb{Q}) \xrightarrow{\mathrm{Res}} \mathrm{Div}^0(X) \longrightarrow 0$$

$w_{D_1} \in T(\mathbb{Q}_p)$ and $\mathrm{Res}(w_{D_1}) = D_1$

Example. (2.14) X hyperelliptic curve $y^2 = f(x)$, $D = P - Q$, where P and Q are non-Weierstrass points. Then

$$w_D = \frac{dx}{2y}\left(\frac{y+y(P)}{x-x(P)} - \frac{y+y(Q)}{x-x(Q}\right) \quad \text{or}$$

$$⚠ \quad w_D = \frac{dx}{2y}\left(\frac{y+y(P)}{x-x(P)} - \frac{y+y(Q)}{x-x(Q}\right) + \eta,$$

where η is a holomorphic differential.

Fix: Move to $J(\mathbb{Q}_p)$

$$T_\ell(\mathbb{Q}_p) = \left\{ \frac{df}{f} \;\middle|\; f \in \mathbb{Q}_p(X)^* \right\} \qquad \left(\mathrm{Res}\left(\tfrac{df}{f}\right) = \mathrm{div}\, f\right)$$

Then, $T_\ell(\mathbb{Q}_p) \cap H^0(X_{\mathbb{Q}_p}, \Omega^1) = 0$ and we get:

$$0 \longrightarrow H^0(X_{\mathbb{Q}_p}, \Omega^1) \longrightarrow T(\mathbb{Q}_p)/T_\ell(\mathbb{Q}_p) \xrightarrow{\mathrm{Res}} J(\mathbb{Q}_p) \longrightarrow 0$$

In set up, we fixed a splitting s of the Hodge filtration on $H^1_{dR}(X/\mathbb{Q}_p)$ such that $\ker(s)$ is isotropic with respect to the cup product pairing.

Let $W := \ker(s) \subseteq H^1_{dR}(X/\mathbb{Q}_p)$.

Then there is a cannonical homomorphism

$$\Psi : T(\mathbb{Q}_p)/T_\ell(\mathbb{Q}_p) \longrightarrow H^1_{dR}(X)$$

such that

(1) Ψ is the identity on differentials of the 1^{st} kind.

(2) Ψ sends 3^{rd} kind differentials to 2^{nd} kind modulo exact differentials

$$\boxed{\text{Given } D \in \mathrm{Div}^0(X),\ \omega_D \text{ is defined as the unique differential of the } 3^{rd} \text{ kind with } \mathrm{Res}(\omega_D) = D \text{ and } \Psi(\omega_D) \in W.}$$

Algorithm (2.22). **Input:** $D_1, D_2 \in \mathrm{Div}^0(X)$

Output: $h_p(D_1, D_2)$

(1) Let $\omega \in T(\mathbb{Q}_p)$ with $\mathrm{Res}(\omega) = D_1$

(2) Compute $\Psi(\omega) = \sum_{i=0}^{2g-1} a_i \omega_i \in H^1_{dR}(X)$

$$\omega_{D_1} = \omega - \sum_{i=1}^{2g-1} a_i \omega_i$$

(Using cup products)

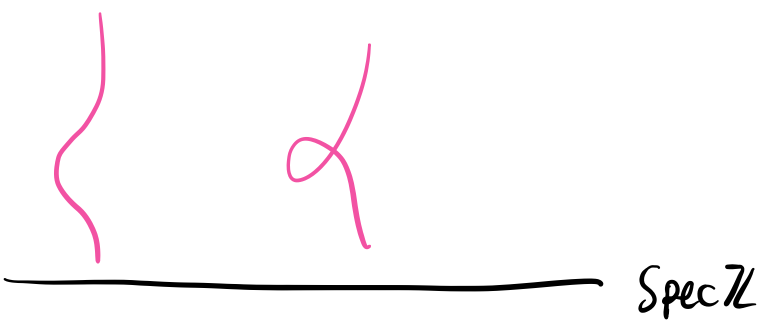
(3) Compute the Coleman integral:

$$\int_{D_2} \omega_{D_1}$$

4. $h_\ell(D_1, D_2)$, $\ell \neq p$.

Given X and $D_1, D_2 \in \mathrm{Div}^0(X)$ with disjoint support, we define $\mathcal{D}_i$ as an extension of D_i to a regular model $\mathbf{X}$ of $X_{\mathbb{Q}_\ell}$ such that $\mathcal{D}_i$ is has trivial intersection with all vertical divisors.
Then

$$h_\ell(D_1, D_2) = \overset{m_\ell}{\underbrace{(\mathcal{D}_1 \cdot \mathcal{D}_2)}_{\substack{\text{Intersection} \\ \text{multiplicity}}}} \log_p(\ell)$$



5=1. Why?

Algorithm. **Input:** $X/\mathbb{Q}$ as in the first theorem.
(2.31 **Output:** The set of integral points on X.

(1) $D_1, \dots, D_g \in \mathrm{Div}^0(X)$ basis for $J(\mathbb{Q}) \otimes \mathbb{Q}$.
Compute $h(D_i, D_j)$.
Another basis for $J(\mathbb{Q}) \otimes \mathbb{Q}$ is:
$\frac{1}{2}(f_k f_\ell + f_\ell f_k)$, where $f_i(P) := \int_0^P \omega_i$

Solve for $\alpha_{k\ell}$:
$$h(D_i, D_j) = \sum_{k,\ell < g-1} \frac{\alpha_{k\ell}}{2}\left(f_k(D_i) f_\ell(D_j) + f_\ell(D_i) f_k(D_j)\right).$$

(2) Compute $\{\overline{\omega_i}\}$ for $0 \le i \le g-1$.
(This is $[\overline{\omega_i}] \cup [\omega_j] = \delta_{ij}$).

(3) Expand $\underset{h_p(z-\infty,\, z-\infty)}{\overset{\shortparallel}{\theta(z)}} := -2 \sum_{i=0}^{g-1} \int \omega_i \overline{\omega_i}$ as a power series

in each residue disk D not containing ∞.

Compute at $\mathbb{Z}_p$-point $P \in D$, $\theta(P)$ and a local coordinate z_P at P

(4) Use intersection theory to compute the finite set S_ℓ of all possible values of

$$h_\ell(z-\infty, z-\infty)$$

for bad primes ℓ and integral $X(\mathbb{Q}_\ell)$.

$\mathbb{Z}$-points!

Obtain a finite set $S \subset \mathbb{Q}_p$ s.t.

$$\sum_{\ell \neq p} h_\ell(P-\infty, P-\infty) \in S \quad \text{for } P \in \mathcal{U}(\mathbb{Z}[\tfrac{1}{p}])$$

(5) Expand $\rho(z) = \theta(z) - \sum_{0 \le i \le j \le g-1} \alpha_{ij} \int_\infty^z \omega_i \int_\infty^z \omega_j$ in

each residue disk, set it equal to each value in S, solve for all $z \in \mathcal{U}(\mathbb{Z}_p)$ such that $\rho(z) \in S$.

Let Z be the collection of such points.