First Homology of Quotients of Fermat Curves

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Outline

1. Overview of the problem
2. Quotients of Fermat curves
3. Homology and modular symbols
4. Describing the classifying element
5. What comes next?

Overview

Let X be a *nice* curve X defined over a number field K.

The étale fundamental group of X is $\pi = [\pi]_1$.

The action of G_K on π gives information on the arithmetic of X.

what about smaller bits?

Overview

Let X be a *nice* curve X defined over a number field K.

geometric

The étale fundamental group of X is $\pi = [\pi]_1$.

The action of G_K on π gives information on the arithmetic of X.

Consider $[\pi]_m = \overline{[\pi, [\pi]_{m-1}]}$.

Proposition [Hain, 97]

There is an isomorphism of G_K-modules

$$[\pi]_2/[\pi]_3 \cong (H_1(X) \wedge H_1(X))/\mathrm{Im}(\mathscr{C}),$$

where $\mathscr{C}$ is the dual map of the cup product map $H_1(X) \wedge H_1(X) \to H_2(X)$.

$$\boxed{V_k : v^p = u(1-u)^k}$$

Let W_k be the projective curve with this affine patch.

Theorem [D. & Pries, 21]

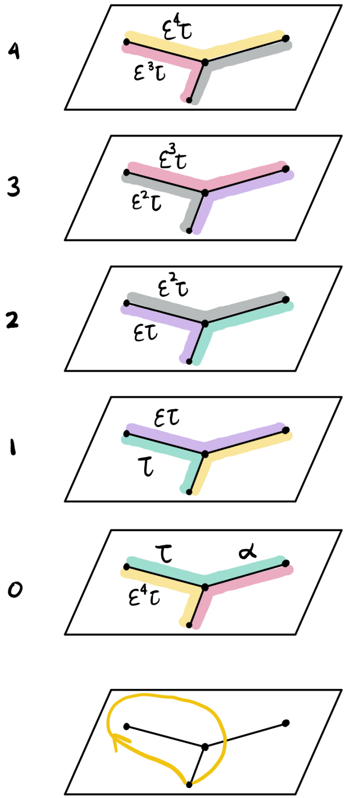
For $p \geq 3$ a prime number and $k \leq p-2$ such that $p \equiv 1 \bmod k+1$, a generator ρ for $\mathrm{Im}(\mathscr{C})$ is given by the image in $H_1(W_k) \wedge H_1(W_k)$ of the following element Δ of $H_1(V_k) \wedge H_1(V_k)$:

$$\Delta = \sum_{1 \leq i < j \leq p-1} c_{i,j} \underbrace{[E_i] \wedge [E_j]}_{\text{explicit}},$$

where

$$c_{i,j} = \begin{cases} 1 & j-i \equiv 0 \bmod k+1 \\ -1 & j-i \equiv 1 \bmod k+1 \\ 0 & \text{otherwise} \end{cases}$$

Example: $y^5 = x(1-x)$



$p=5$

$k=1$

$y^p = x(1-x)$

$$\begin{aligned}\Delta \quad = \quad & E_1 \wedge (-E_2 + E_3 - E_4) \\ & + E_2 \wedge (-E_3 + E_4) \\ & + E_3 \wedge (-E_4).\end{aligned}$$

Definition

Given an integer $n \geq 2$, the **Fermat curve** of degree n is the projective curve given by the equation

$$\boxed{F : X^n + Y^n = Z^n}$$

Note: We fix an odd prime number p. We only consider the Fermat curve of degree p.

Facts we need

- There is a $\mu_p \times \mu_p$ action on F given by

$$(\zeta_p^i, \zeta_p^j) \cdot [X : Y : Z] = [\zeta_p^i X : \zeta_p^j Y : Z]$$

- Let $x = X/Z$ and $y = Y/Z$. For all $r, s \geq 1$ such that $r + s \leq p - 1$, we have that

$$\omega_{r,s} := x^{r-1} y^{s-1} \frac{dx}{y^{n-1}}$$

is a holomorphic one-form on F. Moreover, a basis for the holomorphic one-forms of F is

$$\{\omega_{r,s} \mid r, s \geq 1, r + s \leq p - 1\}.$$

Definition

For $k \in \{1, \ldots, p-2\}$, we let $\boldsymbol{W_k}$ be the quotient of the Fermat curve by the action of $(\zeta_p, \zeta_p^{-k^{-1}}) \in \mu_p \times \mu_p$.

Affine chart:

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Affine chart:

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$V_k : v^p = u(1-u)^k$

$x^p + y^p = 1$

deg p $\left(\begin{array}{l} u = x^p \\ V = xy^k \end{array} \right.$

$v^p = u(1-u)^k$

$\Big\downarrow u$ deg p

$\mathbb{P}^1$

- The genus of W_k is $\dfrac{p-1}{2}$.
- The rank of $H_1(W_k; \mathbb{Z})$ is $p-1$.
- An automorphism of W_k is $\epsilon(u, v) = (u, \zeta_p v)$.
- $W_k \to \mathbb{P}^1$ is a cover of degree p that is branched at 0, 1 and ∞.

Modular Curves

- G is a finite index subgroup of $\mathrm{PSL}_2(\mathbb{Z})$,
- $\tilde{\mathfrak{H}} = \mathfrak{H} \cup \mathbb{P}^1(\mathbb{Q})$,
- for $z \in \tilde{\mathfrak{H}}$ and $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PSL}_2(\mathbb{Z})$, $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot z := \dfrac{az+b}{cz+d}$,
- $X_G(\mathfrak{C}) := \tilde{\mathfrak{H}}/G$ is an irreducible projective algebraic curve,
- $\pi : \tilde{\mathfrak{H}} \to X_G(\mathbb{C})$.

Example [Rohrlich, 77]

$\Phi(p) := \langle A^p, B^p, \Gamma(2)' \rangle$, where $\Gamma(2)'$ is the commutator of $\Gamma(2)$. The modular curve $X_{\Phi(n)}$ is isomorphic to the Fermat curve of degree n.

Recall: W_k is the quotient of F by the automorphism $\left(\zeta_p, \zeta_p^{-(k-1)}\right)$.

Then W_k is isomorphic to X_{Φ_k}, where

$$\Phi_k := \left\langle AB^{-(k-1)},\, A^p,\, B^p,\, \Gamma(2)' \right\rangle .$$

$A \rightsquigarrow [X:Y:Z] \longmapsto [\zeta_p X : Y : Z]$

$B \rightsquigarrow [X:Y:Z] \longmapsto [X : \zeta_p Y : Z]$

Recall: W_k is the quotient of F by the automorphism $\left(\zeta_p, \zeta_p^{-(k^{-1})}\right)$.

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The right cosets of Φ_k in $\mathrm{PSL}_2(\mathbb{Z})$ are:

$$\{[A^r \alpha_j] \mid 0 \leq r \leq p-1, 0 \leq j \leq 5\}.$$

Explicit

Modular Symbols

Definition

Let α, $\beta \in \tilde{\mathfrak{H}}$, the **modular symbol** $\{\alpha, \beta\}$ is the element of $\mathrm{Hom}_{\mathbb{C}}(H^0(X_G(\mathbb{C}), \Omega^1), \mathbb{C})$ given by

$$\{\alpha, \beta\} : \left(\omega \mapsto \int_\alpha^\beta \pi^*\omega\right).$$

Note: We can assume that $\{\alpha, \beta\} \in H_1(X_G(\mathbb{C}), \mathbb{R})$ because of the isomorphism

$$\begin{array}{ccc} H_1(X_G(\mathbb{C}), \mathbb{R}) & \to & \mathrm{Hom}_{\mathbb{C}}(H^0(X_G(\mathbb{C}), \Omega^1), \mathbb{C}) \\ \gamma & \mapsto & \left(\omega \mapsto \int_\gamma \pi^*\omega\right). \end{array}$$

Paths on X_G!

There is an action of $\mathrm{PSL}_2(\mathbb{Z})$ on modular symbols: for $g \in \mathrm{PSL}_2(\mathbb{Z})$,

$$g\{\alpha, \beta\} = \{g \cdot \alpha, g \cdot \beta\}.$$

We let $[g] := g\{0, i\infty\}$.

On X_G, this action is the same on elements of the same right coset

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Lemma

The group of modular symbols for W_k is a free group of rank p, generated by

$$\{[A^r \tau] : 0 \leq r \leq p-1\},$$

where $\tau = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$ and $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$.

First homology of $H_1(W_k, \mathbb{Z})$

Theorem [Manin, 72]

Any class in $H_1(X_G, \mathbb{Z})$ can be represented as a sum

$$\sum_m n_m [g_m]$$

$$\overset{\shortparallel}{g_m\{0, i\infty\}}$$

of modular symbols, where $\sum_m n_m(\pi(g_m \cdot i\infty) - \pi(g_m \cdot 0)) = 0$.

Proposition

The homology group $H_1(W_k, \mathbb{Z})$ is generated by

$$\gamma_r := \underbrace{[\tau] - [A^r\tau]}_{\text{Loops on } W_k} \text{ for } 1 \leq r \leq (p-1).$$

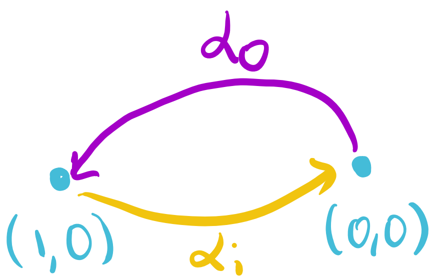
Explicit formulas

Proposition

The relative homology $H_1(W_k, \{0,1\}; \mathbb{Z})$ is generated by

$$\alpha_i = \left(\underset{[0,1]}{t} \mapsto \left(t, \zeta_p^i \sqrt[p]{t(1-t)^k} \right) \right),$$

for $0 \leq i \leq p-1$.



Note: $\varepsilon(x,y) = (x, \zeta_p y)$

$\varepsilon(\alpha_0) = \alpha_1$

$H_1(W_k, \{0,1\}; \mathbb{Z})$ is a free $\mathbb{Z}[\mu_p]$-mod of rank 1!

Construction of Δ

Goal: To find a generator for the image of the map

$$H_2(W_k) \to H_1(W_k) \wedge H_1(W_k),$$

the dual of the cup product.
Why?

$$[\pi]_m = \overline{[\pi, [\pi]_{m-1}]}$$

Theorem, [Labute, 70]

Let F be the free profinite group on $2g$ generators and consider the graded Lie algebra $\mathrm{gr}(\pi) = \oplus_{m \geq 1} [\pi]_m / [\pi]_{m+1}$. Then,

$$\mathrm{gr}(\pi) \cong \mathrm{gr}(F)/\overline{\langle \rho \rangle},$$

where ρ generates the image of $H_2(W_k) \to H_1(W_k) \wedge H_1(W_k)$.

Galois actions

$$\mathrm{gr}(\pi) \cong \mathrm{gr}(F)/\overline{\langle \rho \rangle}.$$

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It suffices to obtain a complete description of the ideal $\langle \rho \rangle$ and the action of μ_p on it.

We have an action on H_1!

Presentation of the fundamental group

Let $U = W_k \setminus \{(0,0),(1,0),\infty\}$ and we choose a base point b as a tangential point to $(0,0)$. There exist loops a_i, b_i for $1 \leq i \leq g$ and c, with base point b, such that $\pi_1(U)$ has a presentation

$$\pi_1(U) = \langle a_i, b_i, c \mid i = 0, \dots, g \rangle / \left(\prod_{i=0}^{g} [a_1, b_i] \right) c$$

We can assume that the loop c circles the punctures $(0,0)$, $(1,0)$ and ∞.

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Fact: The element

$$\Delta = \sum_{i=0}^{g} a_i \wedge b_i$$

generates the kernel of $H_2(U) \to H_1(U) \wedge H_1(U)$.

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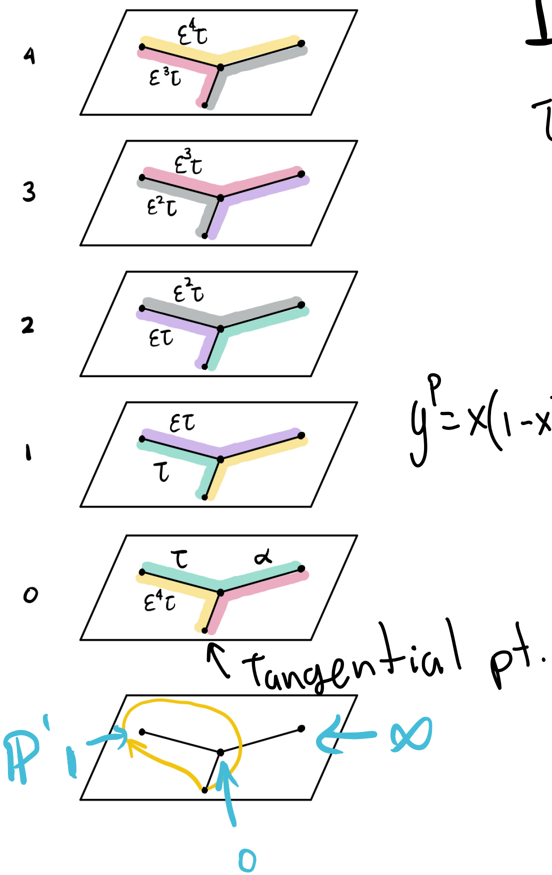

Inertia type: $(1, 1, 3)$

τ $\quad 0 \rightsquigarrow 1$ $\qquad 0 \quad 1 \quad \infty$

$y^p = x(1-x)^k$

$\varepsilon^4\tau$

τ^{-1} $\longrightarrow$

$\varepsilon\tau$

$(\varepsilon^2\tau)^{-1}$

$\varepsilon^3\tau$

$(\varepsilon^4\tau)^{-1}$

τ

$(\varepsilon\tau)^{-1}$

$\varepsilon^2\tau$

$(\varepsilon^3\tau)^{-1}$

E_4

E_1

E_2^{-1}

E_3

E_4^{-1}

E_1^{-1}

E_2

E_3^{-1}

What is next?

- Generalize the formula that we obtained for all $k \leq p - 2$.
- Describe explicitly the action of the absolute Galois group on the homology.
- Can this idea be generalized to other families of curves?

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